Tutte's Spring Theorem

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Abstract: We present a short proof of the theorem of Tutte that every planar 3-connected graph has a drawing in the plane such that every vertex which is not on the outer cycle is the barycenter of its neighbors. Moreover, this holds for any prescribed representation of the outer cycle.
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1. INTRODUCTION

We define a *spring graph* as follows. Let G be a connected graph. We select a cycle C in G, and, for each edge e not in C, we let c_e be a positive real number which we call the *spring constant* of e. We then define a *spring representation* of this spring graph as follows. First, let C' be a representation of C as a convex polygon in the plane. Each other vertex of G is represented by a point in the plane, and each edge is the straight line segment joining its ends. The *energy of the edge* $e = xy$ is defined as $c_e l_e^2$, where l_e is the length of e. The *energy of the representation* of G is the sum of the energies of the edges not in C. As the energy is a convex function of the vertex coordinates, there is precisely one representation which minimizes the energy. This representation is called a spring representation of the spring graph G with respect to the cycle C. If G has a block distinct from the block containing C, then that block is clearly represented by a single point. Therefore we shall assume that G is 2-connected. If G has a set of two vertices u, v such that some component of $G - u - v$ has no vertex in C, then that component is represented by the straight line segment between u and v. Therefore, the 3-connected case is the most interesting.

A *plane representation* of a graph is a drawing of a graph in the plane such that distinct vertices are represented by distinct points, the edges are straight line segments, and two edges intersect only in a common end.

Tutte [1] proved that, if G is planar and 3-connected, and C is any facial cycle (i.e., C is a cycle bounding a face), then any spring representation of G with respect to C is a plane represention of G. This is a representation with nice features: all faces are bounded by convex polygons, and if C is a regular polygon, then any isomorphism of G which takes C to C and preserves spring constants, can be extended to an isometry of the plane.

Tutte's proof of this remarkable result is based on the equations expressing the equilibrium conditions (see below) and makes repeated use of the fact that $K_{3,3}$ is nonplanar. We shall here present an alternative short proof based only on elementary continuity and a reduction lemma for 3-connected graphs.

2. THE SPRING THEOREM

As the energy is a convex function of the coordinates of the vertices, it has only one stationary point. By putting all partial derivatives equal to zero, we conclude that the total force acting on a vertex p of the spring graph but not in C is zero, where the *force* of an edge pq acting on p is the vector from p to q multiplied by the spring constant c_{pq}. This is the *equilibrium condition* at a vertex. Clearly, if the sum of a collection of vectors starting at a vertex p is zero, then no angle between consecutive vectors is greater than 180 degrees. From this follows the following observations: first, no vertices are represented by points outside C. Second, if the spring representation is a plane representation, then all faces are bounded by convex polygons. Finally, if an interior vertex p of a spring graph G has degree 2, and the two edges incident with p have spring constants c_1, c_2, respectively, then the representation of G is also the spring graph representation of the graph obtained from G by replacing p and its two incident edges with one edge with spring constant $c_1c_2/(c_1 + c_2)$.

We shall apply the following slight extension of a reduction lemma due to Barnette and Grünbaum [2].

Lemma 2.1. *Let C be any cycle of a 3-connected graph G distinct from K_4. Then G has an edge e not in C such that $G - e$ is a subdivision of a 3-connected graph unless C has length 3, and G is obtained from C by adding a set of independent vertices each joined to precisely the three vertices of C.*

Proof. A short proof is given in Reference [3]. For the sake of completeness we sketch the idea. We extend successively C to a bigger subgraph such that the current graph is always a subdivision of a 3-connected graph. If C has length 3 and is contained in a K_4, then we first add all those vertices which are joined to all three vertices of C, and we add all edges from these vertices to C. Otherwise, we use Menger's theorem to extend C to a subdivision of K_4. Then we successively add shortest paths to the current graph such that, at each stage, we have a subdivision of a 3-connected graph. If the current graph is 3-connected, we add instead a new vertex and three paths from that vertex to the current graph. If the last step is the addition of a path, then this path has length 1 and can play the role of e. If the last step is the addition of a vertex z of degree 3 with neighbors x_1, x_2, x_3 (say), then we may add $x_i z x_{i+1}$ followed by $z x_{i+2}$ unless $x_i x_{i+1}$ is present in G. If all three edges $x_i x_{i+1}$ are present in G, then we may assume one of them, say $x_j x_{j+1}$, is not an edge of C (since otherwise, z would have been added in the first step) and now $x_j x_{j+1}$ can play the role of e. ■

Theorem 2.1. *Let C be any facial cycle of a 3-connected planar graph G. For each edge e not in C, let c_e be a positive real number. Let C' be a representation of C as a convex polygon in the plane. Let G' be the spring representation of G with spring constants c_e, $e \in E(G) \backslash E(C)$. Then G' is a plane graph.*

Proof. The proof is by induction on the number of edges of G. The theorem is easily verified for K_4 so we proceed to the induction step. As G is planar and C is a facial cycle in G, the last statement of Lemma 2.1 does not hold. By Lemma 2.1, G has an edge e not in C such that $G - e$ is a subdivision of a 3-connected graph H. In other words, G is obtained from H by adding e possibly after subdividing one or two edges. If h is an edge of H which corresponds to two edges i, j in G, then we let $c_i c_j / (c_i + c_j)$ be the spring constant of h. If h is an edge in H which is also an edge in G, then we let c_h be the spring constant of h in H. By the induction hypothesis, the spring representation H' of H is a plane graph. For each positive real number t, let G_t be the spring graph which is isomorphic to G and whose spring constants are the same as those in G except that e has the spring constant t. Let G'_t be the spring representation of G_t. If x is a vertex of G, then we denote by x_t the vertex of G'_t which represents x. For each vertex x in G, x_t is a continuous function of t. Moreover, G'_t converges to H' (with the edge e added) as t tends to 0, and G'_t converges to G' as t tends to c_e. Both of these statements are easy consequences of the uniqueness of the spring representation combined with the fact that if a bounded real function f defined on a real interval has no limit as t tends to t_0, then there exist two sequences of real numbers both tending to t_0 such that the image sequences (under f) tend to distinct numbers.

Now suppose (reductio ad absurdum) that G' is not plane. Then there exists a smallest t_0 such that $G'_{t_0} = M$ is not plane. Informaly, t_0 is the first time G'_t is not a drawing of G.

We claim that t_0 is positive. In other words, if we add e to the plane graph H', then we obtain a plane graph. First, observe that the two ends of e are on the same facial cycle of H'. (They are clearly on the same facial cycle of a plane graph obtained from a plane representation of G by deleting e. But Whitney's theorem on uniquenes of plane representations of 3-connected planar graphs says that the facial cycles are the same in all plane representations of H.) Second, all faces of H' are bounded by convex polygons. Moreover, these polygons are strictly convex in the sense that all angles around a vertex are less than 180 degrees. (Again, this is a consequence of the equilibrium condition mentioned in the "Introduction.") If $G - e$ has two vertices of degree 2, then they must be the ends of e, and therefore they are not on the same edge in H', and so it is possible to add e to H' and preserve a plane representation. Therefore $t_0 \neq 0$.

As M is a finite union of straight line segments, we may think of M as a plane graph (which is not isomorphic to G) whose vertex set consists of all points x_{t_0} where x is a vertex of G. (As edges do not cross we do not introduce new vertices.) Also, all faces of M are bounded by convex polygons, because M is a spring representation. If x is a vertex of G we now write x_M instead of x_{t_0}. Consider two edges xy and uv with no common end in G. The minimality of t_0 implies that the edges $x_M y_M$ and $u_M v_M$ do not cross in M. But, some (or all) of the vertices x_M, y_M, u_M, v_M may coincide. It is also possible that x_M, y_M, u_M are distinct and that u_M is a point on the edge $x_M y_M$. It is also possible that x_M, y_M, u_M, v_M are all distinct and that the edges $x_M y_M$ and $u_M v_M$ intersect and are contained in a common straight line. However, we shall prove that none of these degeneracies occur. We first prove that no interior vertex of G is represented by a vertex of C in M. More precisely:

$$\text{If } c \text{ is a vertex of } C \text{ and } x \text{ is a vertex of } G - C, \text{ then } x_M \neq c_M. \tag{1}$$

Proof of (1). Suppose (reductio ad absurdum) that (1) is false. Let V be the set of vertices x in $G - C$ such that $x_M = c_M$. Since c is not a cutvertex of G, some x in V has a neighbor not in $V \cup \{c\}$. But then the total force acting on x is non-zero (as it is the sum of non-zero vectors all strictly on one side of a line).

This contradiction proves (1). ■

$$\text{If } cd \text{ is an edge of } C \text{ and } x \text{ is a vertex of } G - C, \text{ then } x_M \text{ is not a point on } c_M d_M. \tag{2}$$

Proof of (2). Suppose (reductio ad absurdum) that (2) is false. We repeat the proof of (1) with a minor modification: let V be the set of vertices x in $G - C$ such that x_M is a point on $c_M d_M$. By (1), x_M is distinct from each of c_M, d_M. Since $G - \{c, d\}$ is connected, some x in V has a neighbor not in $V \cup \{c, d\}$. But then the total force acting on x is non-zero.

This contradiction proves (2). ■

$$\text{If } p \text{ is a vertex of degree at least 3 in } M, \text{ then there exists precisely one vertex } x \text{ in } G \text{ such that } x_M = p \tag{3}$$

Proof of (3). Only the uniqueness needs a proof, and by (1) and (2), we may assume that p is in the interior of C. Let V be the set of vertices x in $G - C$ such that $x_M = p$, and suppose (reductio ad absurdum) that V has at least two elements. We claim that there exists a line L through p and there exists some vertex x in V such that all edges of G incident with x are contained in the same closed halfspace of L, and moreover, at least one edge of G incident with x goes into the open halfspace. Once this claim has been established, we obtain a contradiction as in (1) and (2), because then the total force acting on x is non-zero.

So in order to complete the proof of (3) it suffices to prove the claim. Consider first any vertex x in V having a neighbor not in V. As the total force acting on x is zero, there are at least two edges incident with x and going out from V. The edges incident with x and going out from V divide a small disc around p in angular sections. If one of these is strictly greater than 180 degree, then x satisfies the claim. If they are all strictly smaller than 180 degree, then we consider any vertex y in V which is distinct from x and which has a neighbor outside V. (Such a vertex exists because V has at least two elements, and x is not a cutvertex of G.) As no edge incident with y crosses an edge incident with x in G_t when $t < t_0$, it follows that the claim is satisfied with y instead of x. Finally, if two consecutive edges incident with x form an angle of 180 degree in M, then we use the assumption that p has degree at least 3 in M. We consider a third edge g of M incident with p, and we let y be any vertex in V which is incident with g (or, more precisely, y is incident with an edge in G which in M is represented by g). That vertex y satisfies the claim, and the proof of (3) is complete. ■

If u, y, v are vertices in G, such that u_M, y_M, v_M are distinct, uv is an edge of G, and y_M is a point on the edge $u_M v_M$, then y_M has degree 2 in M. (4)

Proof of (4). Suppose (reductio ad absurdum) that (4) is false. We repeat the proof of (3) with a minor modification: let V be the set of vertices x in $G - C$ such that $x_M = y_M$, and let L be the line containing the edge $u_M v_M$. Repeating the argument of (3) completes the proof of (4). ■

We are now ready for the final contradiction. Combining (1–4) with the assumption that M is not a plane representation of G implies that M has at least one vertex of degree 2. In other words, M has a path $P : p_1 p_2 \cdots p_k$, $k \geq 3$, such that p_1, p_k have degree at least 3 in M, and all the intermediate vertices have degree 2 in M. Moreover, the path P is in M represented by a straight line segment which has no point, except possibly the ends, in common with C. By (3), G has precisely one vertex x such that $x_M = p_1$ and precisely one vertex y such that $y_M = p_k$. By (4), G has no edge going through x_M. More precisely, G has no edge uv such that x_M, u_M, v_M are distinct and x_M is a point on the edge $u_M v_M$. A similar statement holds for y_M. But then every path in G from x_2 to C (where x_2 is a vertex represented by p_2) contains one of x, y which contradicts the assumption that G is 3-connected.

This completes the proof. ■

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