Embedding Planar Graphs on the Grid

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Abstract. We show that each plane graph of order $n \geq 3$ has a straight line embedding on the n–2 by n–2 grid. This embedding is computable in time O(n). A nice feature of the vertex-coordinates is that they have a purely combinatorial meaning.

1. Introduction

A *straight line embedding* of a plane graph G is a plane embedding of G in which edges are represented by straight line segments joining their vertices. Theorems of Fáry [F], Stein [St] and Wagner [W] show that each plane graph has straight line embeddings.

As each straight line embedding of a graph is completely specified by giving the positions of its vertices, many algorithms for the construction of these embeddings concentrate uniquely on the determination of the vertex positions.

This paper presents another approach to the problem of constructing straight line embeddings. Its origin can be found in [Sc] where it was used to characterize the planar graphs as graphs whose incidence relation is the intersection of three total orders.

In this approach, rather than directly determining the vertex positions, at first only their relative positions are considered. It is in a second phase that the relative positions are implemented by an actual placement of the vertices.

The actual vertex positions will be expressed in terms of three barycentric coordinates. Therefore, given a plane graph G, the determination of relative positions should result in three partial orders $<_1, <_2, <_3$ on the vertex set of G. These orders will be so defined, that every placement of the vertices in which the vertex coordinates satisfy the conditions $u <_i v \Rightarrow u_i < v_i$ yields a straight line embedding of the graph G.

This method will be applied to the case of *triangular graphs* G (plane graphs whose faces are triangles). In this case, determining $<_1, <_2, <_3$ is equivalent to constructing a partition of the set of interior edges in three trees with special orientation properties. This construction is achieved in linear time. Actual coordinates are then easily computed also in linear time: with each vertex v are associated three regions of G, the coordinates of v are essentially the numbers of vertices (or triangles, edges, ...) in each of the three regions.

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This approach has the following advantages. (1) Straight line embeddings are represented in a coordinate free manner. (2) Both this representation and the actual coordinates are interpreted in terms of structural graph theoretic properties. (3) The resulting embeddings have nice separation properties. For example, the following theorem will directly result from the construction.

THEOREM 1.1. *Let λ_1, λ_2, λ_3 be three pairwise non parallel straight lines in the plane. Then, each plane graph has a straight line embedding in which any two disjoint edges are separated by a straight line parallel to λ_1, λ_2 or λ_3.*

A *straight line embedding on the m by n grid* is a straight line embedding in which vertices v have integer valued (cartesian) coordinates (v_1,v_2) in the range $0 \le v_1 \le m$, $0 \le v_2 \le n$. In relation with the drawing of graphs on finite display devices, it is natural to ask for embeddings on grids of "small" size [**RT**]. Here we show:

THEOREM 1.2. *Each plane graph with $n \ge 3$ vertices has a straight line embedding on the $n-2$ by $n-2$ grid.*

This theorem improves on the previous bound $2n-5$ by $2n-5$ presented in [**Sc**]. It also improves on the bound $n-2$ by $2n-4$ obtained by de Fraysseix, Pach and Pollack [**FPP**], with a different approach.

2. Barycentric representations

DEFINITION. A *barycentric representation* of a graph G is an injective function $v \in V(G) \to (v_1,v_2,v_3) \in \mathbf{R}^3$ that satisfies the conditions:

(1) $v_1 + v_2 + v_3 = 1$ for all vertices v ,

(2) For each edge $\{x,y\}$ and each vertex $z \notin \{x,y\}$, there is some $k \in \{1,2,3\}$ such that $x_k < z_k$ and $y_k < z_k$.

We view v_1, v_2 and v_3 as barycentric coordinates of the vertex v and obtain:

LEMMA 2.1. *Let $v \in V(G) \to (v_1,v_2,v_3)$ be a barycentric representation of a graph G. Then given any three noncolinear points α, β and γ, the mapping $f: v \in V(G) \to v_1\,\alpha + v_2\,\beta + v_3\,\gamma$ is a straight line embedding of G in the plane spanned by α, β and γ.*

PROOF. By definition f is injective. Also, for an edge $\{x,y\}$ and a vertex $z \notin \{x,y\}$ the point f(z) does not lie on the segment f(x)f(y) since the condition $x_k < z_k$ and $y_k < z_k$ must be satisfied for some k.

Suppose now that $\{x,y\}$ and $\{u,v\}$ are disjoint edges. There exist indices $i,j,h,k \in \{1,2,3\}$ satisfying the conditions:

$x_i > u_i,v_i$, $y_j > u_j,v_j$, $u_h > x_h,y_h$, $v_k > x_k,y_k$.

These conditions imply $\{i,j\} \cap \{h,k\} = \emptyset$. As $i,j,h,k \in \{1,2,3\}$ there holds $i = j$ or $h = k$. Therefore the segments f(x)f(y) and f(u)f(v) are separated by a straight line parallel to $\alpha\beta$, $\alpha\gamma$ or $\beta\gamma$ and do not intersect.

Thus only a planar graph G can have barycentric representations and each of these representations *induces* a natural embedding of G in the plane. We consider the plane oriented by requiring that the ordering α, β, γ be counterclockwise.

3. Labeling the angles and interior edges of a triangular graph

Recall that a *plane graph* is an abstract graph together with an embedding of this graph in the plane. A *barycentric representation of a plane graph* G is a barycentric representation f of its underlying abstract graph such that the given embedding of G and the embedding induced by f are equivalent. The next sections will show that each plane graph has a barycentric representation, implying theorem 1.1. We shall use the following terminology.

A *triangular graph* G is a maximal plane graph with at least three vertices. The vertices and edges on the exterior face of G are the *exterior vertices and edges* of G, the other vertices and edges of G are the *interior vertices and edges* of G. The interior faces of G are its *elementary triangles*. The *angles* of G are the angles of its elementary triangles.

If $v \in V(G) \rightarrow (v_1,v_2,v_3) \in \mathbf{R}^3$ is a barycentric representation of a triangular graph G, then each angle $\angle(xy,xz)$ of G determines a unique *label* $k \in \{1,2,3\}$ such that $x_k > y_k,z_k$ (the label is unique since the inequalities $y_j > x_j,z_j$ and $z_i > x_i,y_i$ must also be satisfied). It can be shown that this labeling is a *normal labeling:*

DEFINITION. A *normal labeling* of a triangular graph G is a labeling of the angles of G with the labels 1,2,3 satisfying the conditions:

(1) Each elementary triangle of G has an angle labeled 1, an angle labeled 2 and an angle labeled 3. The corresponding vertices appear in counterclockwise order.

(2) The labels of the angles of each interior vertex v of G form, in counterclockwise order, a nonempty interval of 1's followed by a nonempty interval of 2's followed by a nonempty interval of 3's.

EXAMPLE 3.1. (figure 1).

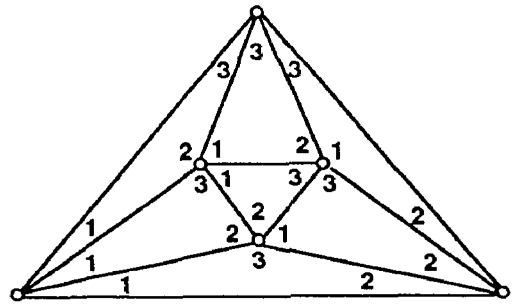


Fig. 1

The following sections will demonstrate that every triangular graph has normal labelings and that each normal labeling is induced by a barycentric representation. Presently, we introduce another equivalent form of labelings: labelings of interior edges.

Consider a normal labeling of a triangular graph G. Each interior edge of G belongs to two elementary triangles and must therefore have at one of its ends two distinct labels $j \neq k$ and at the other end the third label i repeated twice (figure 2a). We call this distinguished i the *label* of the edge and direct this edge from the end with distinct labels to the end with identical labels (figure 2b).

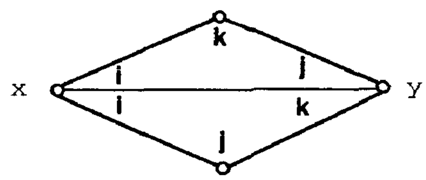


Fig. 2a

Fig. 2b

Note, in particular, that if the normal labeling of G is induced by a barycentric representation, an edge $\{x,y\}$ is labeled with label i and directed from y to x if and only if $y_i < x_i$ and $y_j > x_j$ for $j \neq i$.

EXAMPLE 3.2. In figure 3, the thick edges are the edges labeled with 1.

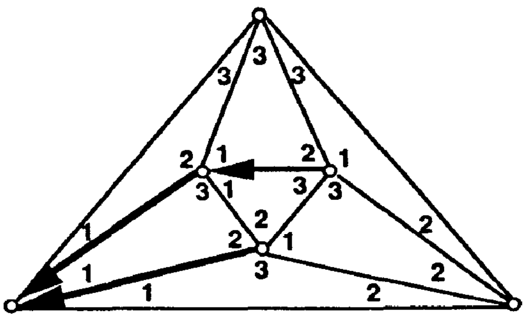


Fig. 3

For $i = 1,2,3$ let T_i consist of all (directed) interior edges having the label i. Condition 2 of the definition of normal labelings implies that T_1, T_2 and T_3 satisfy the following definition.

DEFINITION. A *realizer* of a triangular graph G is a partition of the interior edges of G in three sets T_1, T_2, T_3 of directed edges such that for each interior vertex v there holds:

(1) v has outdegree one in each of T_1, T_2, T_3.

(2) The counterclockwise order of the edges incident on v is: leaving in T_1, entering in T_3, leaving in T_2, entering in T_1, leaving in T_3, entering in T_2.

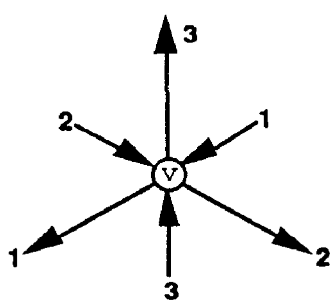


Fig. 4

Thus each normal labeling of a triangular graph induces a realizer of this graph. Conversely, it is easy to see that each realizer of a triangular graph with at least four vertices is induced by a unique normal labeling. Therefore realizers and normal labelings are equivalent notions. We will henceforth simply write *labeled graph* to mean a triangular graph with both a normal labeling and the equivalent realizer.

LEMMA 3.3. *In a labeled triangular graph, all angles at an exterior vertex have the same label and angles at distinct exterior vertices have distinct labels. The exterior vertices whose angles are labeled 1, 2, 3 appear in this order counterclockwise.*

PROOF. Consider a labeled triangular graph with n vertices. This graph has 3n–9 interior edges, directed by the labeling. Define the outdegree of a vertex as the number of interior edges leaving this vertex. Then 3n–9 is the sum of the outdegrees of the vertices. As each of the n–3 interior vertices has outdegree 3, the exterior vertices must have outdegree 0. Each interior edge incident on an exterior vertex is therefore entering this vertex. This implies that all angles at an exterior vertex have the same label. The other statements of the lemma follow from the definition of normal labelings and the fact that each exterior edge belongs to an elementary triangle.

Using first lemma 3.3, then conditions (1) and (2) of the definition of normal labelings, it is easy to manually label all angles of a triangular graph (try it on a large example). The next section shows that this can always be done.

4. Constructing normal labelings

We now prove that each triangular graph has a normal labeling. From this we will deduce in section 6 that each plane graph has barycentric representations. There follows in particular that each plane graph has straight line embeddings, a result of Fáry, Stein and Wagner. Our proof is based on their methods, as analyzed by Kampen [**K**], and will be transformed to a linear time algorithm in section 8.

We first review the method of *edge contraction*. For a vertex x of a graph G, *N(x)* denotes the set of neighbors of x in G. If {x,y} is an edge of G, the *contracted graph G/(x,y)* is obtained from G by removal of the vertex y and the edges incident on y and by insertion of an edge {x,z} for each vertex $z \in N(y)-N(x)$. The edge {x,y} is *contractible* if x and y have exactly two common neighbors. If G is a triangular graph on at least 4 vertices and {x,y} is a contractible edge of G, then G/(x,y) is a triangular graph.

LEMMA 4.1 [**K**]. *Let G be a triangular graph on $n \geq 4$ vertices. If a, b and c denote the exterior vertices of G, then there exists a neighbor $x \neq b,c$ of a such that the edge {a,x} is contractible.*

Thus to each triangular graph corresponds a sequence of "allowed contractions" transforming this graph into a triangle. And, conversely, each triangular graph can be obtained from a triangle by a sequence of "expansions".

THEOREM 4.2. *Each triangular graph has a normal labeling.*

PROOF. We show that given a triangular graph G and an exterior vertex a of G, there is a normal labeling of G in which all angles at a have the label 1. The proof is by induction on the number n of vertices of G. The case $n = 3$ is trivial. Let $n \geq 4$ and assume that our claim is true for all triangular graphs having less than n vertices.

Lemma 4.1 implies the existence of an interior vertex x adjacent to a such that the edge {a,x} is contractible. Let $a, v_1, v_2, \ldots, v_r$ be the vertices of the wheel of x, listed in counterclockwise order.

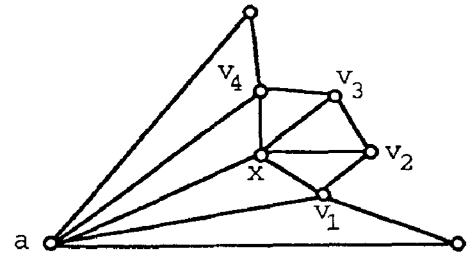


Fig. 5

By induction hypothesis, the graph G/(a,x) has a normal labeling in which all angles at a have the label 1.

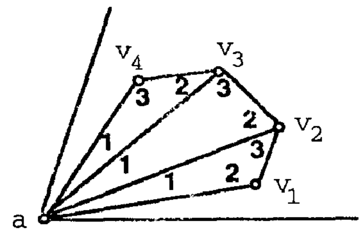


Fig. 6

Without destroying the topology of the labels at $v_1, v_2, \ldots, v_r$ this labeling can be extended to a normal labeling of G by labeling the angles $\angle(xv_i, xv_{i+1})$ and $\angle(av_1, ax)$, $\angle(ax, av_r)$ with 1:

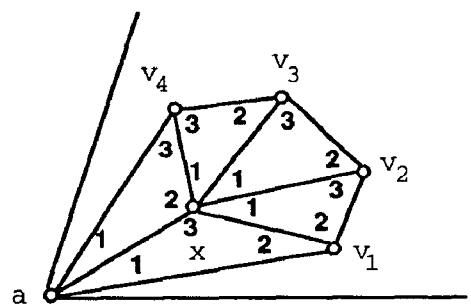


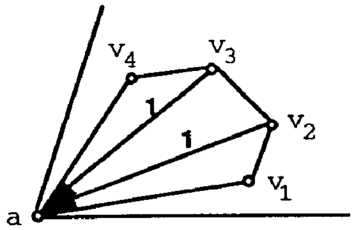
Fig. 7

REMARK 4.3. In terms of realizers, the proof of theorem 4.2 expands a realizer T_1, T_2, T_3 of G/(a,x) to a realizer of G by the operations

$T_1 := T_1 - \{(v_i,a) \mid i \neq 1,r\} \cup \{(x,a)\} \cup \{(v_i,x) \mid i \neq 1,r\}$,

$T_2 := T_2 \cup \{(x,v_1)\}$, $T_3 := T_3 \cup \{(x,v_r)\}$

in which ordered pairs denote directed edges. This is illustrated in figure 8 where only the edges involved have been directed and labeled.



is transformed to

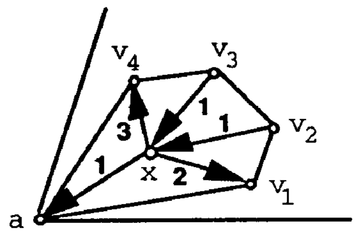


Fig. 8

We claim that the algorithm presented in the proof of theorem 4.2 can generate *every* normal labeling. That is, given a labeled triangular graph G on at least four vertices, it must be verified that: (1) G contains an exterior vertex a all of whose angles have the label 1 and (2) there is a contractible interior edge {a,x} such that the labeling of G results from a labeling of G/(a,x) by the method used in the proof.

The first of these statements was established in lemma 3.3. The second statement follows from lemma 4.4.

LEMMA 4.4. *Let G be a labeled triangular graph with at least four vertices and let a be the exterior vertex of G all of whose angles have the label 1. Then G has an interior vertex x adjacent to a such that {a,x} is contractible and that, except for the angles of the two elementary triangles containing {a,x}, all angles at x have the label 1.*

PROOF. Let $x_0, x_1, \ldots, x_s$ be the counterclockwise list of the neighbors of a in which x_0 and x_s are exterior vertices. (figure 9).

Note that each edge $\{x_i, x_{i+1}\}$ is either labeled with 2 (if directed from x_{i+1} to x_i) or with 3 (if directed from x_i to x_{i+1}). In particular, lemma 3.3 implies that $\{x_0, x_1\}$ and $\{x_{s-1}, x_s\}$ are respectively labeled with 2 and 3. There exists therefore a vertex x_k ($0<k<s$) such that $\{x_{k-1}, x_k\}$ has the label 2 and $\{x_k, x_{k+1}\}$ has the label 3. This implies that except for $\angle(x_k a, x_k x_{k-1})$ and $\angle(x_k x_{k+1}, x_k a)$ all angles at x_k have the label 1.

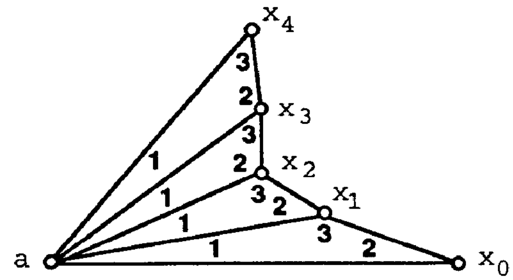


Fig. 9

There remains to verify that $\{a, x_k\}$ is contractible. Suppose on the contrary that some vertex x_i ($i \neq k-1, k+1$) is adjacent to x_k. The topology of the angles at x_k implies that $\{x_i, x_k\}$ is labeled with 1 and directed from x_i to x_k. By lemma 3.3 there holds $i \neq 0, s$. Therefore $\{x_i, a\}$ and $\{x_i, x_k\}$ are two interior edges labeled with 1 and leaving x_i in contradiction to condition 1 of the definition of realizers.

Properties of realizers can therefore be proved by verifying their invariance under the expansions used in the proof of theorem 4.2. The statements of the following theorems 4.5 and 4.6 originate in the interpretation of labelings by barycentric representations.

THEOREM 4.5. *Let G be a triangular graph with at least four vertices and let T_1, T_2, T_3 be a realizer of G. Then each T_i is a tree including all interior vertices and exactly one exterior vertex and all edges of T_i are directed toward this exterior vertex. The exterior vertices belonging to T_1, T_2, T_3 are distinct and appear in counterclockwise order.*

PROOF. This statement clearly holds for a graph with exactly four vertices and remains valid under the expansions described in the proof of theorem 4.2 (see remark 4.3).

We call the exterior vertex belonging to T_i the *root of* T_i. Note that this is the exterior vertex all of whose angles have the label i. We will call this exterior vertex the root of T_i even when G only has three vertices (i.e. $T_1 = T_2 = T_3 = \emptyset$).

THEOREM 4.6. *If* T_1, T_2, T_3 *is a realizer of a triangular graph, then for* $i = 1,2,3$ *the relation* $T_i \cup T_{i+1}^{-1} \cup T_{i+2}^{-1}$ *has no directed cycle (indices are modulo 3).*

PROOF. Due to the cyclic nature of realizers, it suffices to verify this statement for i = 1, i.e., for $T_1 \cup T_2^{-1} \cup T_3^{-1}$. The claim is then trivially true in a triangular graph with three vertices and will remain valid under the expansions described in the proof of theorem 4.2 as the only edge of $T_1 \cup T_2^{-1} \cup T_3^{-1}$ leaving the new vertex x is entering a (see remark 4.3) and a has outdegree 0 in $T_1 \cup T_2^{-1} \cup T_3^{-1}$.

5. The three regions of a vertex

Let G be a labeled triangular graph with realizer T_1, T_2, T_3. Given an interior vertex v of G, we define the *i-path* $P_i(v)$ *starting at* v as the path in T_i from v to the root of T_i. Theorem 4.6 implies that for $i \neq j$, $P_i(v)$ and $P_j(v)$ have v as only common vertex.

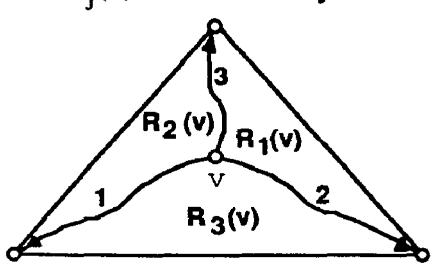


Fig. 10

Therefore, $P_1(v)$, $P_2(v)$ and $P_3(v)$ divide G in three regions $R_1(v)$, $R_2(v)$ and $R_3(v)$ where $R_i(v)$ denotes the closed region opposite to the root of T_i (figure 10).

LEMMA 5.2. *For any two distinct interior vertices u and v of a labeled triangular graph there holds the implication* $u \in R_i(v) \Rightarrow R_i(u) \subset R_i(v)$. *The inclusion is proper.*

PROOF. It suffices to prove the lemma for $i = 3$, thus $u \in R_3(v)$. We only consider the case where u does not lie on the boundary of $R_3(v)$, the other case is similar. Let x denote the first vertex of $P_1(u)$ that belongs to the boundary of $R_3(v)$. Condition 2 of the definition of realizers, applied to the edges incident on x implies that $x \notin P_2(v)$, thus $x \in P_1(v)-\{v\}$. Similarly the first vertex y of $P_2(u)$ belonging to the boundary of $R_3(v)$ must lie on $P_2(v)-\{v\}$. Hence $R_3(u) \subset R_3(v)$. This inclusion is proper as $v \in R_3(v) - R_3(u)$.

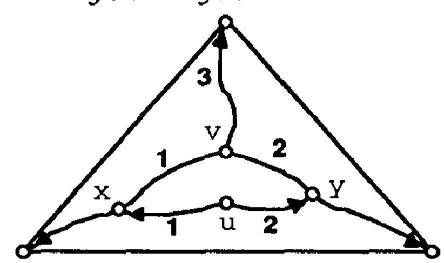


Fig. 11

6. Coordinates that count triangles

In this section we consider a fixed labeled triangular graph G on n vertices with realizer T_1, T_2, T_3. For an interior vertex v of G let v_i be the number of elementary triangles in region $R_i(v)$. For example, in figure 12 we have $(x_1,x_2,x_3) = (1,2,4)$. We extend this definition to the exterior vertices by defining $v_i = 2n-5$, $v_{i+1} = v_{i+2} = 0$ for the root v of T_i (as will always be the case, indices are modulo 3). For example, in figure 12 there holds $(c_1,c_2,c_3) = (0,0,7)$.

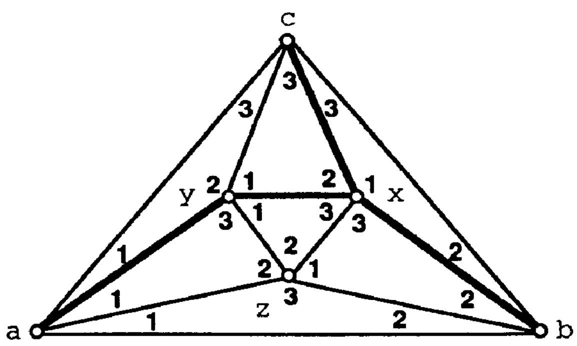


Fig. 12

Notice that $0 \le v_1, v_2, v_3$ and $v_1 + v_2 + v_3 = 2n - 5$ (with $1 \le v_1, v_2, v_3 \le 2n - 7$ if v is an interior vertex).

THEOREM 6.1. *The function* $f : v \in V(G) \to \frac{1}{2n-5}(v_1, v_2, v_3)$ *is a barycentric representation of G and the labeling of G that is induced by f is identical to the given labeling of G.*

PROOF. The first condition of the definition of barycentric representations is clearly satisfied. There remains to verify the second condition (which in turn will imply the injectivity of f). Consider an edge $\{x,y\}$ and a vertex $z \notin \{x,y\}$. If z is an exterior vertex, the root of T_k, there holds $z_k = 2n-5 > x_k, y_k$. Else, z is an interior vertex and $x,y \in R_k(z)$ for some k. This again implies $z_k > x_k, y_k$ (see lemma 5.2).

Furthermore, the labeling induced by f and the given labeling of G are trivially identical if G only has three vertices. If G has four or more vertices it suffices to verify that for each interior edge $\{x,y\}$ labeled with i and directed from y to x there holds $y_i < x_i$ and $y_j > x_j$ for $j \ne i$. This follows from lemma 5.2 since $x \in R_j(y)$ for $j \ne i$.

We may therefore apply lemma 2.1. As the roots a,b,c of T_1, T_2, T_3 are thereby mapped to the reference points α, β, γ, there follows:

COROLLARY 6.2. *Let a, b and c denote the roots of* T_1, T_2 *and* T_3. *Then for any choice of noncolinear positions of a, b and c the mapping*

$$f : v \to \frac{1}{2n-5}(v_1 a + v_2 b + v_3 c)$$

is a straight line embedding of G in the plane spanned by a,b,c.

Choosing, in particular, the gridpoints $a = (2n-5, 0)$, $b = (0, 2n-5)$ and $c = (0,0)$ we obtain:

PROPOSITION 6.3. *The mapping* $v \in V(G) \to (v_1, v_2)$ *is a straight line embedding of G on the 2n–5 by 2n–5 grid.*

REMARK 6.4. The first statement of theorem 6.1 is actually a consequence of its second statement. This explains the description of our method in section 1: the orders $<_1, <_2, <_3$ are the transitive closures of the relations $T_i \cup T_{i+1}^{-1} \cup T_{i+2}^{-1}$.

7. Coordinates that count vertices

More compact layouts will be obtained under relaxation of the constraints imposed on the vertex coordinates by the definition of barycentric representations: we allow for restricted equalities in condition 2 of this definition.

DEFINITION. A *weak barycentric representation* of a graph G is an injective function $v \in V(G) \to (v_1, v_2, v_3) \in \mathbf{R}^3$ that satisfies the conditions:

(1) $v_1 + v_2 + v_3 = 1$ for all vertices v ,

(2) For each edge $\{x,y\}$ and each vertex $z \notin \{x,y\}$, there is some $k \in \{1,2,3\}$ such that $(x_k, x_{k+1}) <_{lex} (z_k, z_{k+1})$ and $(y_k, y_{k+1}) <_{lex} (z_k, z_{k+1})$.

Lemma 2.1 also applies to weak barycentric representations (with a similar proof).

Consider now a fixed labeled triangular graph

G on n vertices with realizer T_1, T_2, T_3. For an interior vertex v of G we let v_i' be the number of vertices in region $R_i(v)$ from which the i–1 path starting at v has been removed, thus $v_i' = |R_i(v)| - |P_{i-1}(v)|$. For example, in figure 12 we have $(x_1', x_2', x_3') = (1,1,3)$. This definition is extended to the exterior vertices of G by setting $v_i' = n-2$, $v_{i+1}' = 1$, $v_{i+2}' = 0$ for the root v of T_i.

For each vertex v there thus holds $v_1' + v_2' + v_3' = n-1$ and $0 \le v_1', v_2', v_3' \le n-2$ (with $1 \le v_1', v_2', v_3' \le n-3$ for the interior vertices).

LEMMA 7.1. *Let u and v be distinct vertices of G. If v is an interior vertex and $u \in R_i(v)$ there holds $(u_i', u_{i+1}') <_{lex} (v_i', v_{i+1}')$.*

PROOF. Notice first that there holds the implication $u \in R_k(v) - P_{k-1}(v) \Rightarrow u_k' < v_k'$. This is clear if u is an exterior vertex as then u is the root of T_{k+1} and $u_k' = 0$ whereas $v_k' \ge 1$ and follows from lemma 5.2 if u is an interior vertex.

Suppose now that $u \in R_i(v)$. Lemma 5.2 implies $u_i' \le v_i'$. If $u \notin P_{i-1}(v)$ we have $u_i' < v_i'$. Else $u \in P_{i-1}(v)$ thus $u \in R_{i+1}(v) - P_i(v)$ and $u_{i+1}' < v_{i+1}'$, by the preceding observation with $k = i+1$.

Lemma 7.1 implies that the function $v \in V(G) \rightarrow (v_1', v_2', v_3')$ is injective and, by an argument similar to the argument used in the proof of theorem 6.1, that $v \in V(G) \rightarrow \frac{1}{n-1}(v_1', v_2', v_3')$ is a weak barycentric representation. Lemma 2.1 can therefore be applied. The choice $\alpha = (n-1, 0)$, $\beta = (0, n-1)$, $\gamma = (0,0)$ in this lemma yields theorem 7.2, from which theorem 1.2 will follow.

THEOREM 7.2. *The mapping $v \in V(G) \rightarrow (v_1', v_2')$ is a straight line embedding of G on the n–2 by n–2 grid.*

8. Description of an embedding algorithm

Given a plane graph with $n \ge 3$ vertices, the calculation of an embedding may be decomposed in three stages:

(1) Determination of a triangular supergraph G of the input graph.

(2) Computation of a realizer T_1, T_2, T_3 of G.

(3) Count of combinatorial objects (vertices, triangles,...) in each region $R_i(v)$ for every vertex v of G.

The first stage can be completed in time O(n) resulting in a n-vertex triangular graph G whose embedding is given by specification of the three exterior vertices a, b, c in counterclockwise order and of a *rotation* for every vertex v (i.e., a list of the neighbors of v in counterclockwise order is available) [**R**].

The second stage is then performed by successive expansions leading from the triangle a, b, c to the full graph G, as described in the proof of theorem 4.2.

To begin, the vertices of G are ordered in an *expansion sequence* $x_1, x_2, ..., x_n$ where $x_1 = b$, $x_2 = c$, $x_n = a$ and $x_3, ..., x_{n-1}$ describes an order in which the interior vertices may successively be inserted. This can be done in time O(n) (see for example [**FPP**], where expansion sequences are called *canonical labelings*).

The trees T_1, T_2, T_3 are then constructed using the operations given in remark 4.3. Notice that, thereby, the vertices $v_1, ..., v_r$ are the neighbors of x that precede x in the expansion sequence and can be found by inspection of the rotation of x. The construction of T_1, T_2, T_3 may first be done in form of three arrays indicating the unique parent of every vertex in each of the trees. This phase requires time O(deg(x)) for each x thus a total time O(n). The arrays are then used, again in time O(n), to create linked lists that will enable traversals of T_1, T_2, T_3 from roots to leaves.

In stage 3, we are given a realizer T_1, T_2, T_3 of G and want to count combinatorial objects. Of particular interest are the quantities:

$p_i(v) := |P_i(v)|$ the number of vertices on the i–path starting at interior vertex v.

$t_i(v) :=$ the number of vertices in the subtree of T_i that is rooted at interior vertex v.

$r_i(v) := |R_i(v)|$ the number of vertices in region $R_i(v)$ for an interior vertex v.

We claim that these quantities can be computed globally (that is for each v and each i) by traversing the trees T_1, T_2, T_3 a constant number of times, thus at a total cost of time O(n). From there follows, in particular, that the function $v \in V(G) \rightarrow (v_1', v_2')$ defined in theorem 7.3 is computable in time O(n) as $v_i' = r_i(v) - p_{i-1}(v)$ for an interior vertex v.

The claim is clearly true for the quantities $p_i(v)$ and $t_i(v)$. Concerning the quantities $r_i(v)$ notice that for each interior vertex $u \in R_i(v)$, the path $P_i(u)$ must intersect $P_{i+1}(v)$ or $P_{i-1}(v)$. Thus u belongs to the subtree of T_i rooted at some vertex $x \in P_{i+1} \cup P_{i-1}$.

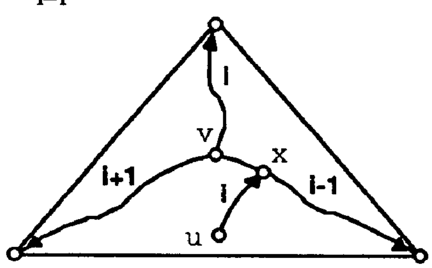


Fig. 13

Furthermore, each of these subtrees is entirely contained in $R_i(v)$. Therefore, extending the definition of $t_i(x)$ to the exterior vertices $x \notin T_i$ by $t_i(x) = 1$, we obtain the expression

$$r_i(v) = \sum_{x \in P_{i+1}(v)} t_i(x) + \sum_{x \in P_{i-1}(v)} t_i(x) - t_i(v)$$

whose sums can be computed by a constant number of traversals of the trees T_1, T_2, T_3.

REMARK 8.1.

(1) Another approach performing stage 2 and a large part of stage 3 simultaneously is also possible as the changes in counting are predictable at each insertion of a new vertex. A related method was used by M. Chrobak and T.H. Payne [CP] in their O(n) implementation of the n–2 by 2n–4 grid embedding.

(2) Stage 2 can be performed in many other ways. For example, the algorithm presented in [R] constructs straight line embeddings in linear time by successively removing vertices of degree at most 5. It may, however, require a very high precision arithmetic. This problem can be avoided by converting the given algorithm to an algorithm for computing realizers (the conversion is easy) and then applying stage 3.

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