# Day 5, SPb SU LOUD Enough Contest 2

February 4, 2023

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- Greedy matching is a matching that maximizes number of edges of weight 1, then number edges of weight 2, etc.
- We get edges in order of increasing of their weights.
- Now, how to solve the problem? Our plan is to solve our problem for edges of weight 1 first and then modify the graph in a way such that solving it for heavier edges won't break anything.

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- Now, we can also see that any maximal matching in the resulting graph could be transformed into a matching having maximal number of light edges.
- Repeat this process for all $q$ weights.

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- Running optimized version of Kuhn at this point of time is a kind of magic that throws against wisdom rather than intelligence, but I'll try to convince you that it clearly fits in TL.

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- Intuitively, the running time of the described algorithm should be $O(\text{Matching}(n, m) + nq)$, but it is hard to prove, as well as it is hard to prove that Kuhn is indeed fast enough for $m = 2 \cdot 10^5$.
- If you are still in doubt, you can implement Dinic' (or Hopcroft–Karp) algorithm with an easy upper bound of $O\left(mq\sqrt{n}\right)$ and a bound of $O\left(m\sqrt{qn}\right)$ that is harder to prove.

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- How do we do that?

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- Thus, we can consider buildings from right to left and each time choose the smallest number of units we should add in order to have enough unassigned shots to destroy the current building.
- So, we solved a problem without shields in time $O\left(n \log \varepsilon^{-1}\right)$. In this problem $\varepsilon \approx 10^{-15}$.

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- Consider the banshee that kills the building. If the building is its first target or was targeted only by it, then the shields never recharge.
- Otherwise, we can exchange its shots for the previous builds with the shots of another banshee that flies further. It contradicts minimality.

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- Proof in one direction is obvious: we can match races to the eliminated players.

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- Note that this proof also provides us an algorithm to construct a permutation from the matching. It is doable even in linear time, if implemented carefully enough.

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- However, there are two key optimizations:
  1. Using Kuhn's algorithm with bitsets. It leads to the complexity of $O\left(\frac{n^4}{w}\right)$.
  2. Reusing a partial matching from one participant to compute answers for another.

  Using any of them could be sufficient to get Accepted, using both of them makes your code fit into a third of TL. Again, you can also use Dinic for better theoretical bounds.

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- Such algorithm is called LateQD, the proof of its optimality is relatively long (but not necessary too hard), it is written in detail in the paper New Competitiveness Bounds for the Shared Memory Switch.
- This algorithm allowed to obtain new lower bounds for competitiveness for the online policy LQD (Longest Queue Drop) which chooses a longest queue whenever is has to push something out.

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- We can store a concatenation of all input queries in a string separately and make nodes of our treap hold views into this string. Now, we don't have to worry about small nodes eating all our memory.
- This is the authors' solution!

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- Now, we will briefly describe some key moments of the solution with sticky braids.

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- In order to do that, we need to remove all unique letters from both strings and then start computing each new row from the cell with a wall. The complexity of this part is $O(|A||B|)$.
- Now, we only need to learn how to concatenate such periodic braids. If we can do that quickly enough, then the problem could be solved using ordinary binary exponentiation.

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- So, we reduced our problem to the known problem of multiplication of sticky braids (sticky multiplication).

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- However, there is an algorithm Steady Ant that does sticky multiplication in time $O(n \log n)$ giving the total complexity for this problem of $O(n \log n \log m + n^2)$.
- The authors are very grateful to Alexander Tiskin for his tremendous help in theoretical side of this problem and also for his great course on string algorithms that was held in SPb SU.