CS 598CSC: Combinatorial Optimization | **Lecture date: 2/16/2009**
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1 Perfect Matching and Matching Polytopes

Let $G = (V, E)$ be a graph. For a set $E' \subseteq E$, let $\chi^{E'}$ denote the characteristic vector of E' in $\mathbb{R}^{|E|}$. We define two polytopes:

$$\mathcal{P}_{perfect\_matching}(G) = \textbf{convexhull}(\{\chi^M | M \text{ is a perfect matching in } G\})$$

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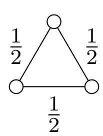
Edmonds gave a description of these polytopes. Recall that for bipartite graphs, $\mathcal{P}_{perfect\_matching}(G)$ is given by

$$\begin{aligned} x\left(\delta(v)\right) &= 1 \quad \forall v \in V \\ x(e) &\geq 0 \quad \forall e \in E \end{aligned}$$

and $\mathcal{P}_{matching}(G)$ is given by

$$\begin{aligned} x\left(\delta(v)\right) &\leq 1 \quad \forall v \in V \\ x(e) &\geq 0 \quad \forall e \in E \end{aligned}$$

We saw an example of a non-bipartite graph, for which $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is an optimum solution.



Hence, (perfect) matching polytope for non-bipartite graphs are not captured by the simple constraints that work for bipartite graphs.

Theorem 1 (Edmonds) *$\mathcal{P}_{perfect\_matching}(G)$ is determined by the following set of inequalities.*

$$\begin{aligned} x(e) &\geq 0; & e \in E \\ x\left(\delta(v)\right) &= 1; & v \in V \\ x\left(\delta(U)\right) &\geq 1; & U \subseteq V, |U| \geq 3, |U| \textit{ odd} \end{aligned}$$

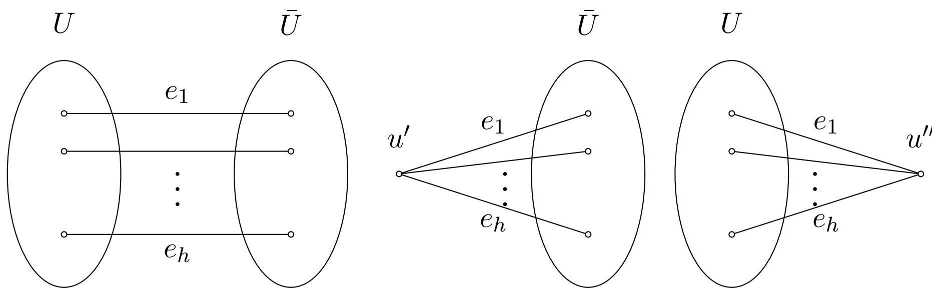
Edmonds gave a proof via an algorithmic method. In particular, he gave a primal-dual algorithm for the minimum cost perfect matching problem, which, as a by product showed that for any cost vector c on the edges, there is a minimum cost perfect matching whose cost is equal the minimum value of cx subjecto to the above set of inequalities. This implies that the polytope is integral. We describe a short non-algorithmic proof that was given later [1] (Chapter 25).

Proof: Let $Q(G)$ denote the polytope described by the inequalities in the theorem statement. It is easy to verify that for each graph G, $\mathcal{P}_{perfect\_matching}(G) \subseteq Q(G)$. Suppose there is a graph G such that $Q(G) \not\subseteq \mathcal{P}_{perfect\_matching}(G)$. Among all such graphs, choose the one that minimizes $|V|+|E|$. Let G be this graph. In particular, there is a basic feasible solution (vertex) x of $Q(G)$ such that x is not in $\mathcal{P}_{perfect\_matching}(G)$.

We claim that $x(e) \in (0,1)$; $\forall e \in E$. If $x(e) = 0$ for some e, then deleting e from G gives a smaller counter example. If $x(e) = 1$ for some e, then deleting e and its end points from G gives a smaller counter example.

We can assume that $|V|$ is even, for otherwise $Q(G) = \emptyset$ (why?) and $\mathcal{P}_{perfect\_matching}(G) = \emptyset$ as well. Since $0 < x(e) < 1$ for each e and $x(\delta(v) = 1$ for all v, we can assume that $deg(v) \geq 2$; $\forall v \in V$. Suppose $|E| = 2|V|$. Then $deg(v) = 2$; $\forall v \in V$ and therefore, G is a collection of vertex disjoint cycles. Then, either G has an odd cycle in its collection of cycles, in which case, $Q(G) = \emptyset = \mathcal{P}_{perfect\_matching}(G)$, or G is a collection of even cycles and, hence bipartite and $Q(G) = \mathcal{P}_{perfect\_matching}(G)$.

Thus $|E| > 2|V|$. Since x is a vertex of $Q(G)$, there are $|E|$ inequalities in the system that are tight and determine x. Therefore, there is some odd set $U \subsetneq V$ such that $x\,(\delta(U)) = 1$. Let $G' = G/U$, where U is shrunk to a node, say u'. Define $G'' = G/\bar{U}$, where $\bar{U} = V - U$ is shrunk to a node u''; see Figure 1. The vector x when restricted to G' induces $x' \in Q(G')$ and similarly x induces



(a) G (b) G' (c) G''

Figure 1:

$x'' \in Q(G'')$. Since G' and G'' are smaller than G, we have that $Q(G') = \mathcal{P}_{perfect\_matching}(G')$ and $Q(G'') = \mathcal{P}_{perfect\_matching}(G'')$. Hence, x' can be written as a convex combination of perfect matchings in G' and x'' can be written as a convex combination of perfect matchings in G''. The vector x is rational since we chose it as a vertex of $Q(G)$, therefore, x', x'' are also rational; hence, $\exists$ integer k such that $x' = \frac{1}{k}\sum_{i=1}^{k}\chi^{M_i'}$, where $M_1', M_2', \cdots, M_k'$ are perfect matchings in G' and $x'' = \frac{1}{k}\sum_{i=1}^{k}\chi^{M_i''}$, where $M_1'', M_2'', \cdots, M_k''$ are perfect matchings in G''. (Note that k is the same in both expressions.)

Let $e_1, e_2, \cdots, e_h$ be edges in $\delta(U)$. Since $x'(\delta(u') = 1$ and u' is in every perfect matching, we have that e_j is in exactly $kx'(e_j) = kx(e_j)$ matchings of $M_1', \cdots, M_k'$. Similarly, e_j is in exactly $kx(e_j)$ matchings of $M_1'', \cdots, M_k''$. Note that $\sum_{j=1}^{n} kx(e_j) = k$ and moreover, exactly one of $e_1, \cdots, e_h$ can be in M_i' and M_i''. We can, therefore, assume (by renumbering if necessary) that M_i' and M_i'' share exactly one edge from $e_1, \cdots, e_h$. Then, $M_i = M_i' \cup M_i''$ is a perfect matching in G. Hence, $x = \frac{1}{k}\sum_{i=1}^{k}\chi^{M_i}$, which implies that $x \in \mathcal{P}_{perfect\_matching}(G)$, contradicting our assumption. $\square$

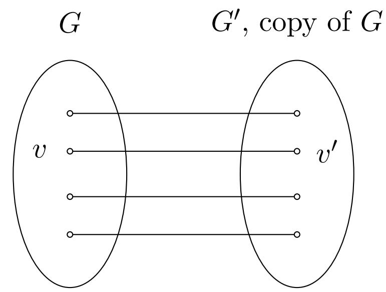
Now, we use the above theorem to derive the following:

Theorem 2 *$\mathcal{P}_{matching}(G)$ is determined by*

$$\begin{aligned} x(e) &\geq 0; & e \in E \\ x(\delta(v)) &\leq 1; & v \in V \\ x(E[U]) &\leq \frac{|U|-1}{2}; & U \subseteq V, |U| \text{ odd} \end{aligned}$$

Here $E[U]$ is the set of edges with both end points in U.

Proof: We can use a reduction of weighted matching to weighted perfect matching that is obtained as follows: Given $G = (V, E)$, create a copy $G' = (V', E')$ of G. And let $\tilde{G}$ be the graph $(\tilde{V}, \tilde{E})$ defined as $\tilde{V} = V \cup V'$, $\tilde{E} = E \cup E' \cup \{(v, v') \mid v \in V\}$.



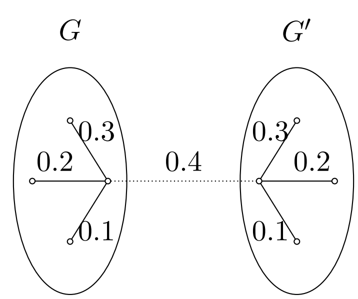
The following claim is easy to prove.

Claim 3 *There is an injective mapping from the matchings in G to the perfect matchings in $\tilde{G}$.*

Corollary 4 *The maximum weight matching problem is poly-time equivalent to maximum weight perfect matching problem.*

We can use the above idea to establish the theorem. Let x be feasible for the system of inequalities in the theorem. We show that x can be written a convex combination of matchings in G. It is clear that χ^M satisfies the inequalities for every matching M. From G, create $\tilde{G}$ as above and define a fractional solution $\tilde{x} : \tilde{E} \to \mathbb{R}^+$ as follows: first, we define $x' : E' \to \mathbb{R}^+$ as the copy of x on E. That is, $x'(e') = x(e)$, where e' is the copy of e. Then,

$$\tilde{x} = \begin{cases} x(e); & \text{if } e \in E \\ x'(e); & \text{if } e \in E' \\ 1 - x(\delta(v)); & \text{if } e = vv' \end{cases}$$

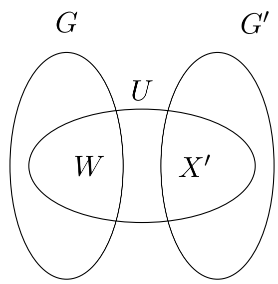


Claim 5 *$\tilde{x}$ belongs to $\mathcal{P}_{perfect\_matching}(\tilde{G})$.*

Assuming the claim, we see that $\tilde{x}$ can be written as a convex combination of perfect matchings in $\tilde{G}$. Each perfect matching in $\tilde{G}$ induces a matching in G and it is easy to verify that x can therefore be written as a convex combination of matchings in G.

It only remains to verify the claim. From the previous theorem, it suffices to show that

$$\tilde{x}\left(\tilde{\delta}(U)\right) \geq 1; \quad \forall U \subseteq \tilde{V}, |U| odd$$

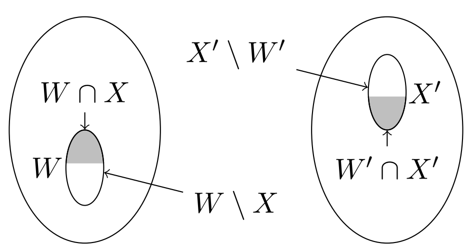


Let $U \subseteq \tilde{V}$ and $|U|$ odd. Let $W = U \cap V$ and $X' = U \cap V'$, where X' is the copy of $X \subseteq V$. First we consider the case that $X' = \emptyset$ and $|W|$ is odd. Then

$$\begin{aligned}
\tilde{x}\left(\tilde{\delta}(U)\right) &= \tilde{x}\left(\tilde{\delta}(W)\right) \\
&= \sum_{v \in W} \tilde{x}\left(\tilde{\delta}(v)\right) - 2\tilde{x}\left(E[W]\right) \\
&= |W| - 2x\left(E[W]\right) \\
&\geq |W| - 2\left(\frac{|W|-1}{2}\right) \\
&\geq 1
\end{aligned}$$

For the general case, we claim that $\tilde{x}\left(\tilde{\delta}(U)\right) \geq \tilde{x}\left(\tilde{\delta}(W \setminus X)\right) + \tilde{x}\left(\tilde{\delta}(X' \setminus W')\right)$. Without loss of generality, $W \setminus X$ is odd. Then $\tilde{x}\left(\tilde{\delta}(U)\right) \geq \tilde{x}\left(\tilde{\delta}(W \setminus X)\right) \geq 1$ from above.

The claim can be verified as follows:



Notice that only edges between W and X' are between $W \cap X$ and $X' \cap W'$. Let $A = W \cap X$, $A' =$

$W' \cap X'$. Then

$$\begin{aligned}
\tilde{x}\left(\tilde{\delta}(U)\right) &= \tilde{x}\left(\tilde{\delta}(W \cup X')\right) \\
&= \tilde{x}\left(\tilde{\delta}(W \setminus X)\right) + \tilde{x}\left(\tilde{\delta}(X' \setminus W')\right) + \\
&\quad x\left(\delta(A)\right) - 2x((\delta\left(E[A, W \setminus X]\right)) + \\
&\quad x\left(\delta'(A')\right) - 2x\left(\delta'\left(E[A', X' \setminus W']\right)\right)
\end{aligned}$$

The claim follows from the observation that $x(\delta(A)) \geq x\left(E[A, W \setminus A]\right) + x\left(\delta(E[A, X \setminus W]\right)$. □

Corollary 6 $\mathcal{P}_{perfect\_matching}(G)$ *is also determined by*

$$\begin{aligned}
x(e) &\geq 0; & e &\in E \\
x\left(\delta(v)\right) &= 1; & v &\in V \\
x\left(E[U]\right) &\leq \frac{|U|-1}{2}; & U &\subseteq V, |U| \text{ odd}
\end{aligned}$$

We note that although the system in the above corollary and the earlier theorem both determine $\mathcal{P}_{perfect\_matching}(G)$, they are not identical.

2 Separation Oracle for Matching Polytope

The inequality systems that we saw for $\mathcal{P}_{perfect\_matching}(G)$ and $\mathcal{P}_{matching}(G)$ have an exponential number of inequalities. Therefore, we cannot use them directly to solve the optimization problems of interest, namely, the maximum weight matching problem or the minimum weight perfect matching problem. To use the Ellipsoid method, we need a polynomial time separation oracle for the polytopes. Edmonds gave efficient strongly polynomial time algorithms for optimizing over these polytopes. From the equivalence of optimization and separation (via the ellipsoid method) implies that there is a strongly polynomial time separation oracle. However, the oracle obtained via the above approach is indirect and cumbersome. Padberg and Rao [1982] gave a simple and direct separation oracle. We discuss this for the system

$$\begin{aligned}
x(e) &\geq 0; & e &\in E \\
x\left(\delta(v)\right) &= 1; & v &\in V \\
x\left(\delta(U)\right) &\geq 1; & |U| &\text{ odd}, U \subseteq V
\end{aligned} \tag{1}$$

and it can be used with the reduction shown earlier for the matching polytope as well.

Theorem 7 *There is a strongly polynomial time algorithm, that given $G = (V, E)$ and $x : E \to \mathbb{R}$ determines if x satisfies (1) or outputs an inequality from (1) that is violated by x.*

It is trivial to check the first two sets of inequalities. Therefore, we assume that $x \geq 0$ and $x\left(\delta(v)\right) = 1; \forall v \in V$. We can also assume that $|V|$ is even. Thus the question is whether there is a set $U \subset V, |U|$ odd, such that $x\left(\delta(U)\right) < 1$. It is sufficient to give an algorithm for the *minimum odd-cut problem*, which is the following: Given a capacitated graph $G = (V, E)$, find a cut $\delta(U)$ of minimum capacity among all sets U such that $|U|$ is odd. We claim that the following is a correct algorithm for the minimum odd-cut problem.

1. Compute a Gomory-Hu tree $T = (V, E_T)$ for G.

2. Among the odd-cuts induced by the edges of T, output the one with the minimum capacity.

To see the correctness of the algorithm, let $\delta(U^*)$ be a minimum capacity odd cut in G. Then $\delta_T(U^*)$ is a set of edges in E_T. We claim that there is an edge $st \in \delta_T(U^*)$ such that $T - st$ has a component with an odd number of nodes. If this is true, then, by the prperties of the Gomory-Hu tree, $T - st$ induces an odd cut in G of capacity equal to $\alpha_G(s,t)$ (recall that $\alpha_G(s,t)$ is the capacity of a minimum s-t cut in G). Since $\delta(U^*)$ separates s and t, the odd cut induced by $T - st$ has no larger capacity than $\delta(U^*)$. We leave it as an exercise to show that some edge in $\delta_T(U^*)$ induces an odd-cut in T.

3 Edge Covers and Matchings

Given $G = (V, E)$ an edge cover is the subset $E' \subset E$ such that each node is covered by some edge in E'. This is the counterpart to vertex cover. Edge covers are closely related to matchings and hence optimization problems related to them are tractable, unlike the vertex cover problem whose minimization version is NP-Hard.

Theorem 8 (Gallai) *Let $\rho(G)$ be the cardinality of a maximum size edge cover in G. Then*

$$\nu(G) + \rho(G) = |V|$$

where $\nu(G)$ is the cardinality of a maximum matching in G.

Proof: Take any matching M in G. Then M covers $2|M|$ nodes, the end points of M. For each such uncovered node, pick an arbitrary edge to cover it. This gives an edge cover of size $\leq |V| - 2|M| + |M| \leq |V| - |M|$. Hence $\rho(G) \leq |V| - \nu(G)$.

We now show that $\nu(G) + \rho(G) \geq |V|$. Let E' be any inclusion-wise minimal edge cover and let M be an inclusion-wise maximal matching in E'. If v is not incident to an edge of M then since it is covered by E' there is an edge $e_v \in E' \setminus M$ that covers v; since M is maximal the other end point of e_v is covered by M. This implies that $2|M| + |E' \setminus M| \geq |V|$, that is $2|M| + |E'| - |M| \geq |V|$ and hence $|M| + |E'| \geq |V|$. If E' is a minimum edge cover then $|E'| = \rho(G)$ and $|M| \leq \nu(G)$, therefore, $\nu(G) + \rho(G) \geq |V|$. □

The above proof gives an efficient algorithm to compute $\rho(G)$ and also a minimum cardinality edge cover via an algorithm for maximum cardinality matching. One can define the minimum weight edge cover problem and show that this also has a polynomial time algorithm by reducing to matching problems/ideas. The following set of inequalities determine the edge cover polytope (the convex hull of the characterstic vectors of edge covers in G).

$$\begin{array}{rcl}
x\left(\delta(V)\right) \geq & 1 & \forall v \in V \\
x\left(E[U] \cup \delta(U)\right) \geq & \frac{|U|+1}{2} & U \subseteq V; |U| odd \\
0 \leq x(e) \leq 1; & & e \in E
\end{array}$$

Exercise 9 *Prove that the polytope above is the edge cover polytope and obtain a polynomial time separation oracle for it.*

References

[1] A. Schrijver. *Combinatorial optimization: polyhedra and efficiency*, Springer, 2003.