# 1 Edmonds-Gallai Decomposition and Factor-Critical Graphs

This material is based on [1], and also borrows from [2] (Chapter 24).

Recall the Tutte-Berge formula for the size of a maximum matching in graph $G$.

**Theorem 1 (Tutte-Berge)** *Given a graph $G$, the size of a maximum cardinality matching in $G$, denoted by $\nu(G)$, is given by:*

$$\nu(G) = \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - o(G - U))$$

*where $o(G-U)$ is the number of connected components in $G[V \setminus U]$ with odd cardinality.*

We call a set $U$ that achieves the minimum on the right hand side of the Tutte-Berge formula, a Tutte-Berge witness set. Such a set $U$ gives some information on the set of maximum matchings in $G$. In particular we have the following.

- All nodes in $U$ are covered in every maximum matching of $G$.
- If $K$ is the vertex set of a component of $G-U$, then every maximum matching in $G$ covers at least $\lfloor K/2 \rfloor$ nodes in $K$. In particular, every node in an even component is covered by every maximum matching.

A graph can have different Tutte-Berge witness sets as the example in Fig 1 shows. Clearly $U = \{v\}$ is more useful than $U = \emptyset$.

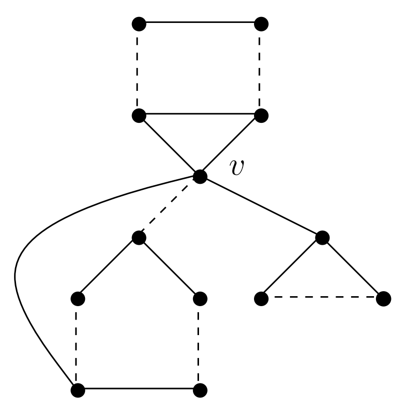


Figure 1: Graph $G$ above has 13 nodes, and $\nu(G) = 6$. $U = \emptyset$ and $U = \{v\}$ are both Tutte-Berge witness sets.

A natural question is whether each graph has a *canonical* Tutte-Berge witness set that gives us as much information as possible. The content of the Edmonds-Gallai decomposition is to give a description of such a canonical witness set. Before we give the theorem, we digress to describe some other settings with canonical witness sets. The reader can skip the next paragraph.

Let $D = (V, A)$ be a directed graph and $s, t \in V$. It is easy to see that $s$ has no directed path to $t$ iff there is a set $X \subseteq V$ such that $s \in X$, $t \notin X$ and $\delta^+(X) = \emptyset$, that is no arcs leave $X$. Among all such sets $X$, the set $X^*$ defined as the set of all nodes reachable from $s$, is a canonical set. It is a simultaneous witness for all nodes that are not reachable from $s$. Moreover, most algorithms for checking reachability of $t$ from $s$ would output $X^*$. Similarly, consider the $s$-$t$ maximum flow problem in a capacitated directed graph $D$. By the maxflow-mincut theorem, the maximum flow value $F$ is equal to the capacity of a minimum capacity cut $\delta(X)$ that separates $s$ from $t$. Again, there could be multiple minimum cuts. One can show that if $\delta(X)$ and $\delta(Y)$ are $s$-$t$ minimum cuts (here $X$ and $Y$ contain $s$ and do not contain $t$) then $\delta(X \cap Y)$ and $\delta(X \cup Y)$ are also minimum cuts (follows from submodularity of the cut function). From this, it follows that there exists a unique minimal minimum cut $\delta(X^*)$ and a unique maximal minimum cut $\delta(Y^*)$. We note that $X^*$ is precisely the set of vertices reachable from $s$ in the residual graph of *any* maximum flow; similarly $V \setminus Y^*$ is the set of nodes that can reach $t$ in the residual graph of any maximum flow.

**Factor-Critical Graphs:** If $U$ is a non-empty Tutte-Berge witness set for a graph $G$, then it follows that there are nodes in $G$ that are covered in every maximum matching.

**Definition 2** *A graph $G = (V, E)$ is* factor-critical *if $G$ has no perfect matching but for each $v \in V$, $G - v$ has a perfect matching.*

Factor-critical graphs are connected and have an odd number of vertices. Simple examples include odd cycles and the complete graph on an odd number of vertices.

**Theorem 3** *A graph $G$ is factor-critical if and only if for each node $v$ there is a maximum matching that misses $v$.*

**Proof:** If $G$ is factor-critical then $G - v$ has a perfect matching and hence a maximum matching in $G$. We saw the converse direction in the proof of the Tutte-Berge formula — it was shown that if each node $v$ is missed by some maximum matching then $G$ has a matching of size $(|V| - 1)/2$. □

If $G$ is factor-critical then $U = \emptyset$ is the unique Tutte-Berge witness set for $G$ for otherwise there would be a node that is in every maximum matching. In fact the converse is also true, but is not obvious. It is an easy consequence of the Edmonds-Gallai decomposition to be seen shortly. We give a useful fact about factor-critical graphs.

**Proposition 4** *Let $C$ be an odd cycle in $G$. If the graph $G/C$, obtained by shrinking $C$ into a single vertex, is factor-critical then $G$ is factor-critical.*

**Proof Sketch.** Let $c$ denote the vertex in $G/C$ in place of the shrunken cycle $C$. Let $v$ be an arbitrary node in $V(G)$. We need to show that $G - v$ has a perfect matching.

If $v \notin C$ then $G/C - v$ has a perfect matching $M$ that matches $c$, say via edge $cu$. When we unshrink $c$ into $C$, let $w$ be the vertex in $C$ that corresponds to the edge $cu$. We can extend $M$ a perfect matching in $G - v$ by adding edges in the even length path $C - w$ to cover all the nodes in $C - w$.

If $v \in C$, consider a perfect matching $M$ in $G/C - c$. It is again easy to extend $M$ to a perfect matching in $G - v$ by considering $C - v$. □

We will see later a structural characterization of factor-critical graphs via ear decompositions.

### 1.1 Edmonds-Gallai Decomposition

**Theorem 5 (Edmonds-Gallai)** *Given a graph $G = (V, E)$, let*

$$\begin{aligned} D(G) &:= \{v \in V \mid \textit{there exists a maximum matching that misses } v\} \\ A(G) &:= \{v \in V \mid v \textit{ is a neighbor of } D(G) \textit{ but } v \notin D(G)\} \\ C(G) &:= V \setminus (D(G) \cup A(G)). \end{aligned}$$

*Then, the following hold.*

1. *The set $U = A(G)$ is a Tutte-Berge witness set for $G$.*
2. *$C(G)$ is the union of the even components of $G - A(G)$.*
3. *$D(G)$ is the union of the odd components of $G - A(G)$.*
4. *Each component in $G - A(G)$ is factor-critical.*

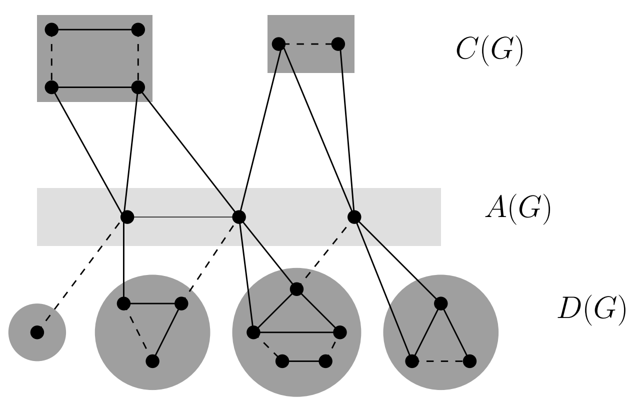


Figure 2: Edmonds-Gallai Decomposition

**Corollary 6** *A graph $G$ is factor-critical if and only if $U = \emptyset$ is the unique Tutte-Berge witness set for $G$.*

We prove the theorem in the rest of this section. We make use of the properties, and proof of correctness, of Edmonds algorithm for maximum cardinality matching that we discussed in the previous lecture.

Let $M$ be any maximum matching in $G$ and let $X$ be the $M$-exposed nodes. We define three sets of nodes with respect to $M$ and $X$.

$$\begin{aligned} \textsc{Even}(G, M) &:= \{v \in V \mid \text{there is an even length } M\text{-alternating } X\text{-}v \text{ path}\} \\ \textsc{Odd}(G, M) &:= \{v \in V \mid \text{there is an } M\text{-alternating } X\text{-}v \text{ path}\} \setminus \textsc{Even} \\ \textsc{Free}(G, M) &:= \{v \in V \mid \text{there is no } M\text{-alternating } X\text{-}v \text{ path}\} \end{aligned}$$

Note that $v \in \textsc{Odd}(G, M)$ implies that there is an odd length $M$-alternating $X$-$v$ path but no even length path. A node $v \in \textsc{Even}(G, M)$ may have both an even and odd length path; also $X \subseteq \textsc{Even}(G, M)$.

**Lemma 7** *For any maximum matching $M$ in $G$ we have (i)* $\textsc{Even}(G, M) = D(G)$ *(ii)* $\textsc{Odd}(G, M) = A(G)$ *and (iii)* $\textsc{Free}(G, M) = C(G)$.

**Proof:** We prove the claims in order. If $v \in \textsc{Even}(G, M)$, let $P$ be an even length $M$-alternating path from some $x \in X$ to $v$. Then, $M \Delta E(P)$ is another maximum matching in which $v$ is exposed; hence, $v \in D(G)$. Conversely, if $v \in D(G)$ there is a maximum matching $M_v$ that misses $v$. Then $M \Delta M_v$ gives an even length $X$-$v$ $M$-alternating path implying that $v \in \textsc{Even}(G, M)$. Therefore, $\textsc{Even}(G, M) = D(G)$.

If $v \in \textsc{Odd}(G, M)$, let $P$ be an $X$-$v$ $M$-alternating path. Since $v \notin \textsc{Even}(G, M)$, $P$ is of odd length and its last edge is $uv$ where $u \in \textsc{Even}(G, M)$. Therefore $v$ is a neighbor of $\textsc{Even}(G, M) = D(G)$ and $v \notin D(G)$ and hence $v \in A(G)$. Conversely, suppose $v \in A(G)$ and let $uv \in E$ where $u \in D(G) = \textsc{Even}(G, M)$. There is an $M$-alternating $X$-$u$ path $P$ of even length which ends in an edge $wu \in M$. If $v \in V(P)$ then clearly there is an $X$-$v$ alternating path. Otherwise, $P + uv$ is an $X$-$v$ alternating path ($wu \in M$, hence $uv \notin M$ unless $w = v$ but then $v \in V(P)$). Therefore $v \in \textsc{Odd}(G, M)$ since $v \notin D(G) = \textsc{Even}(G, M)$.

Finally, $C(G) = V \setminus (D(G) \cup A(G))$ and hence $\textsc{Free}(G, M) = C(G)$. □

**Lemma 8** *Let $M$ be any maximum matching in $G$, then each node in $A(G) \cup C(G)$ is covered by $M$ and moreover every node $v \in A(G)$ is matched to some node in $D(G)$.*

**Proof:** From Lemma 7, $X \subseteq D(G)$ where $X$ is the set of $M$-exposed nodes. Hence each node in $A(G) \cup C(G)$ is covered by $M$.

Suppose $u \in A(G)$ and $uv \in M$. Since $u \in \textsc{Odd}(G, M)$, there is an odd length $X$-$v$ alternating path $P$ which ends in an edge $wu \notin M$. If $v$ is not in $P$ then $P + uv$ is an $M$-alternating $X$-$v$ path and hence $v \in \textsc{Even}(G, M) = D(G)$. If $v$ is in $P$, let $Q$ be the prefix of $P$ till $v$, then $Q + vu$ is an even length $M$-alternating $X$-$u$ path which contradicts the fact that $u \in A(G)$. □

**Corollary 9** *Each component in $G[C(G)]$ is even and $|M \cap C(G)| = |C(G)|/2$.*

**Proof:** All nodes in $C(G)$ are covered by $M$. Since $A(G)$ separates $D(G)$ from $C(G)$, and $A(G)$ is matched only to $D(G)$ (by the above lemma), nodes in $C(G)$ are matched internally and hence the corollary follows. □

The main technical lemma is the following.

**Lemma 10** *Let $M$ be a maximum matching in $G$ and $X$ be the $M$-exposed nodes. Each component $H$ of $G[D(G)]$ satisfies the following properties:*

1. *Either $|V(H) \cap X| = 1$ and $|M \cap \delta_G(V(H))| = 0$, or $|M \cap \delta_G(V(H))| = 1$.*
2. *$H$ is factor-critical.*

Assuming the above lemma, we finish the proof of the theorem. Since each component of $G[D(G)]$ is factor-critical, it is necessarily odd. Hence, from Corollary 9 and Lemma 10, we have that $G[C(G)]$ contains all the even components of $G - A(G)$ and $G[D(G)]$ contains all the odd components of $G - A(G)$. We only need to show that $A(G)$ is a Tutte-Berge witness. To see this, consider any maximum matching $M$ and the $M$-exposed nodes $X$. We need to show $|M| = \frac{1}{2}(|V| + |A(G)| - o(G - A(G)))$. Since $|M| = \frac{1}{2}(|V| - |X|)$, this is equivalent to showing that $|X| + |A(G)| = o(G - A(G))$. From Lemma 8, $M$ matches each node in $A(G)$ to a node in $D(G)$. From Lemma 10, each odd component in $G[D(G)]$ either has a node in $X$ and no $M$-edge to $A(G)$ or has exactly one $M$-edge to $A(G)$. Hence $|X| + |A(G)| = o(G - A(G))$ since all the odd components in $G - A(G)$ are in $G[D(G)]$.

We need the following proposition before the proof of Lemma 10.

**Proposition 11** *Let $M$ be a maximum matching in $G$. If there is an edge $uv \in G$ such that $u, v \in \textsc{Even}(G, M)$, then there is an $M$-flower in $G$.*

**Proof Sketch.** Let $P$ and $Q$ be even length $M$-alternating paths from $X$ to $u$ and $v$, respectively. If $uv \neq M$ then $P + uv + Q$ is an $X$-$X$ alternating *walk* of odd length; since $M$ is maximum, this walk has an $M$-flower. If $uv \in M$, then $uv$ is the last edge of both $P$ and $Q$ and in this case $P - uv + Q$ is again an $X$-$X$ alternating walk of odd length. □

**Proof of Lemma 10.** We proceed by induction on $|V|$. Let $M$ be a maximum matching in $G$ and $X$ be the $M$-exposed nodes. First, suppose $D(G)$ is a stable set (independent set). In this case, each component in $G[D(G)]$ is a singleton node and the lemma is trivially true.

If $G[D(G)]$ is not a stable set, by Proposition 11, there is an $M$-flower in $G$. Let $B$ the $M$-blossom with the node $b$ as the base of the stem. Recall that $b$ has an even length $M$-alternating path from some node $x \in X$; by going around the odd cycle according to required parity, it can be seen that $B \subseteq \textsc{Even}(G, M) = D(G)$. Let $G' = G/B$ be the graph obtained by shrinking $B$. We identify the shrunken node with $b$. Recall from the proof of correctness of Edmonds algorithm that $M' = M/B$ is a maximum matching in $G'$. Moreover, the set of $M'$-exposed nodes in $G'$ is also $X$ (note that we identified the shrunken node with $b$, the base of the stem, which belong to $X$ if the stem consists only of $b$). We claim the following with an informal proof.

**Claim 12** *$D(G') = (D(G) \setminus B) \cup \{b\}$, and $A(G') = A(G)$ and $C(G') = C(G)$.*

**Proof Sketch.** We observed that $X$ is the set of exposed nodes for both $M$ and $M'$. We claim that $v \in \textsc{Even}(G', M')$ implies $v \in \textsc{Even}(G, M)$. Let $P$ be an even length $X$-$v$ $M'$-alternating path in $G'$. If it does not contain $b$ then it is also an $X$-$v$ even length $M$-alternating path in $G$. If $P$ contains $b$, then one can obtain an even length $X$-$v$ $M$-alternating path $Q$ in $G$ by expanding $b$ into $B$ and using the odd cycle $B$ according to the desired parity. Conversely, let $v \in \textsc{Even}(G, M) \setminus B$ and let $P$ be an $X$-$v$ $M$-alternating path of even length in $G$. One can obtain an even length $X$-$v$ $M'$-alternating path $Q$ in $G'$ as follows. If $P$ does not intersect $B$ then $Q = P$ suffices. Otherwise, we consider the first and last nodes of $P \cap B$ and shortcut $P$ between them using the necessary parity by using the odd cycle $B$ and the matching edges in there. Therefore, $D(G') = (D(G) \setminus B) \cup \{b\}$ and the other claims follow. □

By induction, the components of $G' - A(G')$ satisfy the desired properties. Except for the component $H_b$ that contains $b$, every other such component is also a component in $G - A(G)$. Therefore, it is not hard to see that it is sufficient to verify the statement for the component $H$ in $G - A(G)$ that contains $B$ which corresponds to $H_b$ in $G' - A(G')$ that contains $b$. We note that

$X$ is also the set of $M'$-exposed nodes in $G'$ and since $\delta_G(H) \cap M = \delta_{G'}(H_b) \cap M'$ ($B$ is internally matched by $M$ except possibly for $b$), the first desired property is easily verified.

It remains to verify that $H$ is factor-critical. By induction, $H_b$ is factor-critical. Since $H_b$ is obtained by shrinking an odd cycle in $H$, Proposition 4 show that $H$ is factor-critical. □

**Algorithmic aspect:** Given $G$, its Edmonds-Gallai decomposition can be efficiently computed by noting that one only needs to determine $D(G)$. A node $v$ is in $D(G)$ iff $\nu(G) = \nu(G - v)$ and hence one can use the maximum matching algorithm to determine this. However, as the above proof shows, one can compute $D(G)$ in the same time it takes to find $\nu(G)$ via the algorithm of Edmonds, which has an $O(n^3)$ implementation. The proof also shows that given a maximum matching $M$, $D(G)$ can be obtained in $O(n^2)$ time.

## 1.2 Ear Decompositions and Factor-Critical Graphs

A graph $H$ is obtained by *adding an ear* to $G$ if $H$ is obtained by adding to $G$ a path $P$ that connects two not-necessarily distinct nodes $u, v$ in $G$. The path $P$ is called an ear. $P$ is a *proper* ear if $u, v$ are distinct. An ear is an *odd* (even) ear if the length of $P$ is odd (even). A sequence of graph $G_0, G_1, \ldots, G_k = G$ is an ear decomposition for $G$ starting with $G_0$ if for each $1 \leq i \leq k$, $G_i$ is obtained from $G_{i-1}$ by adding an ear. One defines, similarly, proper ear decomposition and odd ear decomposition by restricting the ears to be proper and odd respectively.

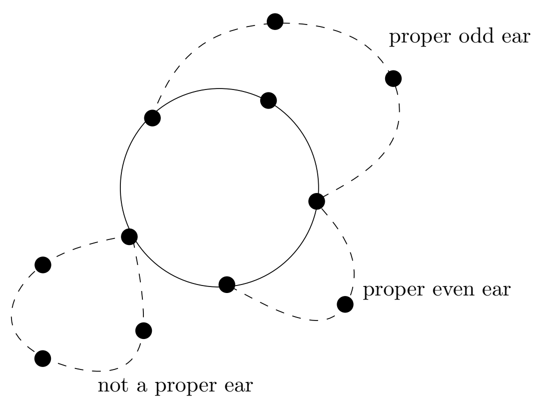


Figure 3: Variety of ears.

The following theorems are well-known and standard in graph theory.

**Theorem 13 (Robbins, 1939)** *A graph $G$ is 2-edge-connected if and only if it has an ear-decomposition starting with a cycle.*

**Theorem 14 (Whitney, 1932)** *A graph $G$ is 2-node-connected if and only if it has a proper ear-decomposition starting with a cycle.*

Factor-critical graphs have the following characterization.

**Theorem 15 (Lovász, 1972)** *A graph $G$ is factor-critical if and only if it has an odd ear decomposition starting with a single vertex.*

**Proof:** If $G$ has an odd ear decomposition it is factor-critical by inductively using Proposition 4 and noting that an odd cycle is factor-critical for the base case.

We now prove the converse. $G$ is necessarily connected. Let $v$ be an arbitrary vertex and let $M_v$ be a perfect matching in $G - v$. We iteratively build the ear decomposition starting with the empty graph $v$. At each step we maintain a (edge-induced) subgraph $H$ of $G$ such that $H$ has an odd ear decomposition and no edge $uv \in M_v$ crosses $H$ (that is, $|V(H) \cap \{u, v\}| \neq 1$). The process stops when $E(H) = E(G)$. Suppose $E(H) \neq E(G)$, then since $G$ is connected, there is some edge $ab \in E(G)$ such that $a \in V(H)$ and $b \neq V(H)$. By the invariant, $ab \notin M_v$. Let $M_b$ be a perfect matching in $G$ that misses $b$. Then $M_b \Delta M_v$ contains an even length $M_v$-alternating path $Q := u_0 = b, u_1, \ldots, u_t = v$ starting at $b$ and ending at $v$. Let $j$ be the smallest index such that $u_j \in V(H)$ ($j$ exists since $u_t = v$ belongs to $V(H)$); that is $u_j$ is the first vertex in $H$ that the path $Q$ hits starting from $b$. Then, by the invariant, $u_{j-1}u_j \notin M_v$ and hence $j$ is even. The path $a, b = u_0, u_1, \ldots, u_j$ is of odd length and is a valid ear to add to $H$ while maintaining the invariant. This enlarges $H$ and hence we eventually reach $G$ and the process generates an odd ear decomposition. □

One can extend the above proof to show that $G$ is 2-node-connected and factor-critical iff it has an proper odd ear decomposition starting from an odd cycle.

From Proposition 4 and Theorem 15, one obtains the following.

**Corollary 16** *$G$ is factor-critical iff there is an odd cycle $C$ in $G$ such that $G/C$ is factor-critical.*

# References

[1] Lecture notes from Michel Goemans class on Combinatorial Optimization. `http://math.mit.edu/~goemans/18438/lec3.pdf`, 2009.

[2] A. Schrijver. *Theory of Linear and Integer Programming (Paperback)*. Wiley, 1998.