Material taken mostly from [1] (Chapter 19).

1 Integer Decomposition Property

A polyhedron P has the *integer decomposition property* if $\forall$ integers $k \geq 1$ and $x \in P$, kx is integral implies $kx = x_1 + x_2 + \ldots + x_k$ for integral vectors $x_1, \ldots, x_k$ in P. Baum and Trotter showed the following:

Theorem 1 (Baum and Trotter) *A matrix A is TUM iff $P = \{x \mid x \geq 0, Ax \leq b\}$ has the integer decomposition property for all integral verctors b.*

Proof: We show one direction, the one useful for applications. Suppose A is TUM, consider $P = \{x \mid x \geq 0, Ax \leq b\}$. Let $y = kx^*$ be an integral vector where $x^* \in P$. We prove by induction on k that $y = x_1 + x_2 + \ldots . x_k$ for integral vectors $x_1, x_2, \ldots, x_k$ in P.

Base case for $k = 1$ is trivial.

For $k \geq 2$, consider the polyhedron $P' = \{x \mid 0 \leq x \leq y; Ay - kb + b \leq Ax \leq b\}$. P' is an integral polyhedron since A is TUM and $Ay - kb + b$ and b are integral. The vector $x^* \in P'$ and hence P' is not empty. Hence there is an integral vector $x_1 \in P'$. Moreover $y' = y - x_1$ is integral and $y' \geq 0$, $Ay' \leq (k-1)b$.

By induction $y' = x_2 + \ldots + x_{k-1}$ where $x_2, \ldots, x_{k-1}$ are integral vectors in P. $y = x_1 + \ldots + x_k$ is the desired combination for y. $\square$

Remark 2 *A polyhedron P may have the integer decomposition property even if the constraint matrix A is not TUM. The point about TUM matrices is that the property holds for all integral right hand side vectors b.*

2 Applications of TUM Matrices

We saw that network matrices are TUM and that some matrices arising from graphs are network matrices. TUM matrices give rise to integral polyhedra, and in particular, simultaneously to the primal and dual in the following when A is TUM and c, b are integral vectors.

$$\max\{cx \mid x \geq 0, Ax \leq b\} = \min\{yb \mid y \geq 0, yA \geq c\}$$

We can derive some min-max results and algorithms as a consequence.

2.1 Bipartite Graph Matchings

Let $G = (V, E)$ be a bipartite graph with $V = V_1 \oplus V_2$ as the bipartition. We can write an integer program for the maximum cardinality matching problem as

$$\begin{array}{rcl} \max & & \sum_{e \in E} x(e) \\ x(\delta(u)) & \leq & 1 \ \forall u \in V \\ x(e) & \geq & 0 \ \forall e \in E \\ x & \in & \mathbb{Z} \end{array}$$

We observe that this is a ILP problem $\max\{1 \cdot x \mid Mx \leq 1, x \geq 0, x \in \mathbb{Z}\}$ where M is the edge-vertex incidence matrix of G. Since M is TUM, we can drop the integrality constraint and solve the *linear program* $\max\{1 \cdot x \mid Mx \leq 1, x \geq 0\}$ since $\{x \mid Mx \leq 1, x \geq 0\}$ is an integral polyhedron. The dual of the above LP is

$$\begin{array}{rcl} \min \sum_{u \in V} y(u) & & \\ y(u) + y(v) & \geq & 1 \quad uv \in E \\ y & \geq & 0 \end{array}$$

in other words $\min\{y \cdot 1 | yM \geq 1, y \geq 0\}$ which is is also an integral polyhedron since M^T is TUM. We note that this is the min-cardinality vertex cover problem

Note that the primal LP is a polytope and hence has a finite optimum solution. By duality, and integrality of the polyhedra, we get that both primal and dual have integral optimum solutions x^* and y^* such that $1 \cdot x^* = y^* \cdot 1$. We get as an immediate corollary König's Theorem.

Theorem 3 *In a bipartite graph the cardinality of a maximum matching is equal to the cardinality of a minimum vertex cover.*

Also, by poly-time solvability of linear programming, there is a polynomial time algorithm for maximum matching and minimum vertex cover in bipartite graphs, and also their weighted versions. Note that we have much more efficient combinatorial algorithms for these problems. We also obtain that $\{x \mid Mx \leq 1, x \geq 0\}$ is the convex hull of the characteristic vectors of the matchings in G and that $\{x \mid Mx = 1, x \geq 0\}$ is the convex hull of the perfect matchings of G.

One easy consequence is the following theorem

Theorem 4 (Birkhoff - Von Neumann) *Let A be a $n \times n$ doubly stochastic matrix. Then A can be written as a convex combination of permutation matrices.*

A doubly stochastic matrix is a square non-negative matrix in which each row and column sum is 1. A permutation matrix is a square $\{0, 1\}$ matrix that has a single 1 in each row and column. Each permulation matrix corresponds to a permutation σ in S_n, the set of all permutations on an n-element set.

Exercise 5 *Prove the above theorem using the perfect matching polytope description for bipartite graphs. How many permutations matrices do you need in the convex combination?*

Note that $\max\{wx \mid Mx \leq 1, x \geq 0\} = \min\{y \cdot 1 \mid yM \geq w, y \geq 0\}$ has integer primal and dual solutions x^* and $y^* \cdot 1$ if w is integral. For the primal x^* corresponds to a maximum w-weight matching. In the dual, via complementary slackness, we have

$$y^*(u) + y^*(v) = w^*(v)$$

for all $x^*(uv) > 0$ Interpreting $y^*(u)$ as a weight on u, one obtains a generalization of König's theorem, known as the Egervary theorem.

Another theorem on bipartite graphs is the Hall's marriage theorem.

Theorem 6 (Hall) *Let $G = (V, E)$ be a bipartite graph with X, Y as the vertex sets of the bipartition. Then there is a matching that saturates X iff $|N(S)| \geq |S| \forall S \subseteq X$ where $N(S)$ is the set of neighbors of S.*

Exercise 7 *Derive Hall's theorem from König's Theorem.*

A generalization of the above also holds.

Theorem 8 *Let $G = (X \cup Y, E)$ be a bipartite graph. Let $R \subseteq U$. Then there is a matching that covers R iff there exists a matching M that covers $R \cap X$ and a matching that covers $R \cap Y$. Therefore, a matching covers R iff $|N(S)| \geq |S| \forall S \subseteq R \cap X$ and $\forall S \subseteq R \cap Y$.*

Exercise 9 *Prove above theorem.*

b-matchings: b-matchings generalize matchings. Given an integral vector $b : V \rightarrow \mathbb{Z}^+$, a b-matching is a set of edges such that the number of edges incident to a vertex v is at most $b(v)$. From the fact that the matrix M is TUM, one can obtain various properties of b-matchings by observing that the polyhedron

$$\begin{aligned} Mx &\leq b \\ x &\geq 0 \end{aligned}$$

is integral for integral b.

2.2 Single Commodity Flows and Cuts

We can derive various useful and known facts about single commodity flows and cuts using the fact that the directed graph arc-vertex incidence matrix is TUM.

Consider the s-t maximum-flow problem in a directed graph $D = (V, A)$ with capacities $c : A \rightarrow \mathbb{R}^+$. We can express the maximum flow problem as an LP with variables $x(a)$ for flow on arc a.

$$\begin{aligned} \max \sum_{a \in \delta^+(s)} x(a) - \sum_{a \in \delta^-(s)} x(a) & & \\ \sum_{a \in \delta^+(v)} x(a) - \sum_{a \in \delta^-(v)} x(a) &= 0 \quad \forall v \in V - \{s, t\} \\ x(a) &\leq c(a) \quad \forall a \in A \\ x(a) &\geq 0 \quad \forall a \in A \end{aligned}$$

Note that the polyhedron defined by the above is of the form $\{x \mid M'x = 0, 0 \leq x \leq c\}$ where M' is the arc-vertex incidence matrix of D with the columns corresponding to s, t removed. M' is a submatrix of M, the arc-vertex incidence matrix of D which is TUM, and hence M' is also TUM. Therefore, the polyhedron above is integral for integral c. One immediate corollary is that for integral capacities, there is a maximum flow that is integral. We now derive the maxflow-mincut theorem as a consequence of the total unimodularity of M.

The dual to the maximum-flow LP above has two sets of variables. $y(a), a \in A$ for the capacity constraints and $z(v), v \in V - \{s,t\}$ for the flow conservation constraints. We let $w(a)$ be the weight vector of the primal. Note that

$$w(a) = \begin{cases} 1 & \text{if } a = (s,v) \text{ for some } v \in V \\ -1 & \text{if } a = (v,s) \text{ for some } v \in V \\ 0 & \text{otherwise} \end{cases}$$

For simplicity assume that there is no arc (s,t) or (t,s). Then the dual is:

$$\begin{array}{rcll} \min \sum_{a \in A} c(a)y(a) & & & \\ z(u) - z(v) + y(u,v) & \geq & 0 & (u,v) \in A \quad \{u,v\} \cap \{s,t\} = \emptyset \\ z(v) + y(s,v) & \geq & 1 & \forall (s,v) \in A \\ z(v) + y(s,v) & \geq & -1 & \forall (v,s) \in A \\ z(v) + y(v,t) & \geq & 0 & \forall (v,t) \in A \\ -z(v) + y(t,v) & \geq & 0 & \forall (t,v) \in A \\ y & \geq & 0 & \end{array}$$

Note that z are unconstrained variables. In matrix form, the primal is $\max\{wx \mid M'x = 0, 0 \leq x \leq c\}$ and the dual is $\min\{yc \mid y \geq 0; \exists z : y + zM' \geq w^T\}$. Since w is integral and M' is TUM, dual is an integral polyhedron. Primal is bounded polyhedron and hence primal and dual have optimal solution x^* and (y^*, z^*) such that $wx^* = y^*c$ and y^*, z^* is integral.

We can extend z to have variables $z(s)$ and $z(t)$ with $z(s) = -1$ and $z(t) = 0$. Then the dual has a cleaner form, $\max\{yc \mid y \geq 0, \exists z : y + zM \geq 0\}$. Note that M here is the full arc-vertex incidence matrix of D. Thus we have x^* and integral (y^*, z^*) such that $wx^* = y^*c$ and $y^* + z^*M \geq 0$.

Let $U = \{v \in V | z^*(v) < 0\}$. Note that $s \in U$ and $t \notin U$ and hence $\delta^+(U)$ is a s-t cut.

Claim 10 $c(\delta^+(U)) \leq y^*c = \sum_{a \in A} y^*(a)c(a)$

Proof: Take any arc $(u,v) \in \delta^+(U)$. We have $z^*(u) - z^*(v) + y^*(u,v) \geq 0$ for each (u,v). Since $u \in U$ and $v \notin U$, $z^*(u) < 0$ and $z^*(v) \geq 0$. Since z^* is integral, we have

$$\begin{array}{rcl} y^*(u,v) & \geq & 1 \\ \Longrightarrow c(\delta^+(U)) & \leq & \sum_{a \in \delta^+(U)} c(a)y^*(a) \\ & \leq & \sum_{a \in A} c(a)y^*(a) \text{ since } y^* \geq 0 \end{array}$$

$\square$

Therefore, U is a s-t cut of capacity at most $y^*c = wx^*$ but wx^* is the value of a maximum flow. Since the capacity of any cut upper bounds the maximum flow, we have that there exists a cut of capacity equal to that of the maximum flow. We therefore, get the following theorem,

Theorem 11 *In any directed graph $G = (V, A)$ with non-negative arc capacities, $c : E \to \mathbb{Q}^+$, the s-t maximum-flow value is equal to the s-t minimum cut capacity. Moreover, if $c : E \to F^+$, then there is an integral maximum flow.*

Interpretation of the dual values: A natural interpretation of the dual is the following. The dual values, $y(a)$ indicate whether a is cut or not. The value $z(v)$ is the shortest path distance from s to v with $y(a)$ values as the length on the arcs. We want to separate s from t. So, we have (implicitly) $z(s) = -1$ and $z(t) = 0$. The constraints $z(u) - z(v) + y(u, v) \geq 0$ enforce that the z values are indeed shortest path distances. The objective function $\sum_{a \in A} c(a)y(a)$ is the capacity of the cut subject to separating s from t.

Circulations and lower and upper bounds on arcs: More general applications of flows are obtained by considering both lower and upper bounds on the flow on arcs. In these settings, circulations are more convenient and natural.

Definition 12 *For a directed graph $D = (V, A)$, a circulation is a function $f : A \to \mathbb{R}^+$ such that $\sum_{a \in \delta^-(v)} f(a) = \sum_{a \in \delta^+(v)} f(a) \forall v \in V$*

Given non-negative lower and upper bounds on the arcs, $l : A \to \mathbb{R}^+$ and $u : A \to \mathbb{R}^+$, we are interested in circulations that satisfy the bounds on the arcs. In other words, the feasibility of the following:

$$\begin{gathered} l(a) \leq x(a) \leq u(a) \\ x \text{ is a circulation} \end{gathered}$$

The above polyhedron is same as $\{x \mid Mx = 0, l \leq x \leq u\}$ where M is the arc-vertex incidence graph of D, which is TUM. Therefore, if l, u are integral then the polyhedron is integral. Checking if there is a feasible circulation in a graph with given l and u is at least as hard as solving the maximum flow problem.

Exercise 13 *Given D, $s, t \in V$ and a flow value F, show that checking if there is an $s - t$ flow of value F can be efficently reduced to checking if a given directed graph has a circulation respecting lower and upper bounds.*

The converse is also true however; one can reduce circulation problems to regular maximum-flow problems, though it takes a bit of work.

Min-cost circulation is the problem: $\min\{cx \mid l \leq x \leq u, Mx = 0\}$. We therefore obtain that

Theorem 14 *The min-cost circulation problem with lower and upper bounds can be solved in (strongly) polynomial time. Moreover, if l, u are integral then there exists an integral optimum solution.*

The analogue of max flow-min cut theorem in the circulation setting is Hoffman's circulation theorem.

Theorem 15 *Given $D = (V, A)$ and $l : A \to \mathbb{R}^+$ and $u : A \to \mathbb{R}^+$, there is a feasible circulation $x : A \to \mathbb{R}^+$ iff*

1. *$l(a) \leq c(a)\ \forall a \in A$ and*
2. *$\forall U \subseteq V, l(\delta^-(U)) \leq c(\delta^+(U))$.*

Moreover, if l, u are integral then there is an integral circulation.

Exercise 16 *Prove Hoffman's theorem using TUM property of M and duality.*

b-Transshipments: One obtains slightly more general objects called transshipments as follows:

Definition 17 *Let $D = (V, E)$ be a directed graph and $b : A \to \mathbb{R}$. A b-transshipment is a function $f : A \to \mathbb{R}^+$ such that $\forall u \in V$, $f(\delta^-(u)) - f(\delta^+(u)) = b(u)$ i.e, the excess inflow at u is equal to $b(u)$.*

We think of nodes u with $b(u) < 0$ as supply nodes and $b(u) > 0$ as demand nodes. Note that $b = 0$ captures circulations. Once can generalize Hoffman's circulation theorem.

Theorem 18 *Given $D = (V, A)$, $b : V \to \mathbb{R}^+$ and $l : A \to \mathbb{R}^+$and $u : A \to \mathbb{R}^+$,there exists a b-transshipment respecting l, u iff*

1. *$l(a) \leq u(a)\ \forall a \in A$ and*
2. *$\sum_{v \in V} b(v) = 0$ and*
3. *$\forall S \subseteq V$, $u(\delta^+(S)) \geq l(\delta^-(S)) + b(S)$.*

Moreover, if b, l, u are integral, there is an integral b-transshipment.

Exercise 19 *Derive the above theorem from Hoffman's circulation theorem.*

2.3 Interval graphs

A graph $G = (V, E)$ on n nodes is an interval graph if there exist a collection $\mathcal{I}$ of n closed intervals on the real line and a bijection $f : V \to \mathcal{I}$ such that $uv \in E$ iff $f(u)$ and $f(v)$ intersect. Given an interval graph, an interesting problem is to find a maximum weight independent set in G where $w : A \to \mathbb{R}^+$ is a weight function. This is same as asking for the maximum weight non-overlapping set of intervals in a collection of intervals.

We can write an LP for it. Let $\mathcal{I} = \{I_1, \ldots, I_n\}$

$$
\begin{aligned}
\max \sum_{i=1}^{n} w_i x_i & & & \\
\sum_{I_i : p \in I_i} x_i & \leq 1 & & \forall p \text{ in } \mathbb{R} \\
x_i & \geq 0 & & 1 \leq i \leq n
\end{aligned}
$$

Note that the constraints can be written only for a finite set of points which correspond to the end points of the intervals. These are the natural "clique" constraints: each maximal clique in G corresponds to a point p and all the intervals containing p. and clearly an independent set cannot pick more than one node from a clique.

The LP above is $\max\{wx \mid x \geq 0, Mx \leq 1\}$ where M is a consecutive ones matrix, and hence TUM. Therefore, the polyhedron is integral. We therefore have a polynomial time algorithm for the max-weight independent set problem in interval graphs. This problem can be easily solved efficienctly via dynamic programming. However, we observe that we can also solve $\max\{wx|x \geq 0, Mx \leq b\}$for any integer b and this is not easy to see via other methods.

To illustrate the use of integer decomposition properties of polyhedra, we derive a simple and well known fact.

Proposition 20 *Suppose we have a collection of intervals $\mathcal{I}$ such that $\forall p \in \mathbb{R}$ the maximum number of intervals containing p is at most k. Then $\mathcal{I}$ can be partitioned into $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k$ such that each $\mathcal{I}_k$ is a collection of non-overlapping intervals. In other words, if G is an interval graph then $\omega(G) = \chi(G)$ where $\omega(G)$ is the clique-number of G and $\chi(G)$ is the chromatic number of G.*

One can prove the above easily via a greedy algorithm. We can also derive this by considering the independent set polytope $\{x \mid x \geq 0, Mx \leq 1\}$. We note that $x^* = \frac{1}{k}.\overline{1}$ is feasible for this polytope if no point p is contained in more than k intervals. Since P has the integer decomposition property, $y = kx^* = \overline{1}$ can be written as $x_1 + \ldots + x_k$ where x_i is integral $\forall 1 \leq i \leq k$ and $x_i \in P$. This gives the desired decomposition. The advantage of the polyhedral approach that one obtains a more general theorem by using an arbitrary integral b in the polytope $\{x \mid x \geq 0, Mx \leq b\}$ and this has applications; see for example [2].

References

[1] A. Schrijver. *Theory of Linear and Integer Programming (Paperback)*. Wiley, 1998.

[2] P. Winkler and L. Zhang. Wavelength Assignment and Generalized Interval Graph Coloring. ACM-SIAM SODA, 2003.