1 Integer Programming, Integer Polyhedra, and Totally Unimodular Matrices

Many discrete optimization problems are naturally modeled as an integer (linear) programming (ILP) problem. An ILP problem is of the form

$$\begin{aligned} &\max cx \\ &Ax \leq b \\ &x \text{ is an integer vector.} \end{aligned} \tag{1}$$

It is easy to show that ILP is NP-hard via a reduction from say SAT. The decision version of ILP is the following: Given rational matrix A and rational vector b, does $Ax \leq b$ have an integral solution x?

Theorem 1 *Decision version of ILP is in NP, and hence it is NP-Complete.*

The above theorem requires technical work to show that if there is an integer vector in $Ax \leq b$ then there is one whose size is polynomial in $size(A, b)$.

A special case of interest is when the number of variables, n, is a fixed constant but the number of constraints, m, is part of the input. The following theorem is known.

Theorem 2 (Lenstra's Algorithm) *For each fixed n, there is a polynomial time algorithm for ILP in n variables.*

1.1 Integer Polyhedra

Given a rational polyhedron $P = \{x|Ax \leq b\}$, we use P_I to denote the convex hull of all the integer vectors in P; this is called the integer hull of P.

It is easy to see that if P is a polytope then P_I is also a polytope. Somewhat more involved is the following.

Theorem 3 *For any rational polyhedron P, P_I is also a polyhedron.*

Definition 4 *A rational polyhedron P is an integer polyhedron if and only if $P = P_I$.*

Theorem 5 *The following are equivalent:*

(i) $P = P_I$ *i.e., P is integer polyhedron.*

(ii) Every face of P has an integer vector.

(iii) Every minimal face of P has an integer vector.

(iv) $\max\{cx|x \in P\}$ *is attained by an integer vector when the optimum value is finite.*

Proof: (i)$\Longrightarrow$(ii): Let F be a face, then $F = P \cap H$, where H is a supporting hyperplane, and let $x \in F$. From $P = P_I$, x is a convex combination of integral points in P, which must belong to H and thus to F.
(ii)$\Longrightarrow$(iii): it is direct from (ii).
(iii)$\Longrightarrow$(iv): Let $\delta = \max\{cx : x \in P\} < +\infty$, then $F = \{x \in P : cx = \delta\}$ is a face of P, which has an integer vector from (iii).
(iv)$\Longrightarrow$(i): Suppose there is a vector $y \in P \setminus P_I$. Then there is an inequality $\alpha x \leq \beta$ valid for P_I while $\alpha y > \beta$ (a hyperplane separating y and P_I). It follows that $\max\{\alpha x \mid x \in P_I\} \leq \beta$ while $\max\{\alpha x \mid x \in P\} > \beta$ since $y \in P \setminus P_I$. Then (iv) is violated for $c = \alpha$. $\square$

Another useful theorem that characterizes integral polyhedra, in full generality due to Edmons and Giles [1977], is the following.

Theorem 6 *A rational polyhedron P is integral if and only if* $\max\{cx \mid Ax \leq b\}$ *is an integer for each integral vector c for which the maximum is finite.*

1.2 Totally Unimodular Matrices

Totally Unimodular Matrices give rise to integer polyhedra with several fundamental applications in combinatorial optimization.

Definition 7 *A matrix A is totally unimodular (TUM) if the determinant of each square submatrix of A is in $\{0, 1, -1\}$. In particular, each entry of A is in $\{0, 1, -1\}$.*

Proposition 8 *If A is TUM and U is a non-singular square submatrix of A, then U^{-1} is an integral matrix.*

Proof: $U^{-1} = \dfrac{U^*}{det(U)}$ where U^* is the adjoint matrix of U. From the definition of total unimodularity, U^* only contains entries in $\{0, +1, -1\}$ and $det(U) = 1$ or -1. Therefore, U is an integral matrix. $\square$

Theorem 9 *If A is TUM then for all integral b, the polyhedron $P = \{x|Ax \leq b\}$ is an integer polyhedron.*

Proof: Consider any minimal face F of P. $F = \{x|A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$. Let A' have $m' \leq n$ rows. Then $A' = [U \quad V]$, where U is a $m' \times m'$ matrix of full row and column rank (after potentially rearranging rows and columns). U is a submatrix of A and hence $det(U) \in \{-1, +1\}$. Therefore $A'x = b'$ has an integer solution $\begin{pmatrix} U^{-1}b' \\ 0 \end{pmatrix}$. Thus every minimal face has an integer vector and hence P is an integer polyhedron. $\square$

We claim several important corollaries.

Corollary 10 *If A is TUM then for all integral vector a, b, c, d, the polyhedron $\{x|a \leq x \leq b, c \leq Ax \leq d\}$ is integral.*

Proof: If A is TUM, so is the matrix $\begin{bmatrix} I \\ -I \\ A \\ -A \end{bmatrix}$. This can be easily proven by expanding the submatrix along the row associated with the identity matrix. $\square$

Proposition 11 *A is TUM $\iff A^T$ is TUM.*

Corollary 12 *If A is TUM and b, c are integral vectors, then $\max\{cx | Ax \leq b, x \geq 0\} = \min\{yb | yA \leq c, y \geq 0\}$ are attained by integral vectors x^* and y^*, if they are finite.*

Proof: The polyhedron $\{y | y \geq 0, yA \leq c\}$ is integral since A^T is TUM and also $\begin{bmatrix} A^T \\ -I \end{bmatrix}$. $\square$

There are many characterizations of TUM matrices. We give a few useful ones below. See [1] (Chapter 19) for a proof.

Theorem 13 *Let A be a matrix with entries in $\{0, +1, -1\}$. Then the followings are equivalent.*

(i) A is TUM.

(ii) For all integral vector b, $\{x | Ax \leq b, x \geq 0\}$ has only integral vertices.

(iii) For all integral vectors a, b, c, d, $\{x | a \leq x \leq b, c \leq Ax \leq d\}$ has only integral vertices.

(iv) Each collection of column S of A can be split into two sets S_1 and S_2 such that the sum of columns in S_1 minus the sum of columns in S_2 is a vector with entries in $\{0, +1, -1\}$.

(v) Each nonsingular submatrix of A has a row with an odd number of nonzero components.

(vi) No square submatrix of A has determinant $+2$ or -2.

$(i) \iff (ii)$ is the Hoffman-Kruskal's theorem. $(ii) \implies (iii)$ follows from the fact that A is TUM $\implies \begin{bmatrix} I \\ -I \\ A \\ -A \end{bmatrix}$ is TUM. $(i) \iff (iv)$ is Ghouila-Houri's theorem.

Several important matrices that arise in combinatorial optimization are TUM.

Example 1: Bipartite Graphs. Let $G = (V, E)$ an undirected graph. Let M be the $\{0, 1\}$ edge-vetex incidence matrix defined as follows. M has $|E|$ rows, one for each edge and $|V|$ columns, one for each vertex. $M_{e,v} = 1$ if e is incident to v otherwise it is 0. The claim is that M is TUM iff G is bipartite.

To see bipartiteness is needed, consider the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ for a triangle which is an odd cycle. Its determinant is 2.

Exercise 1 *Show that edge-vertex adjacency matrix of any odd cycle has determinant 2.*

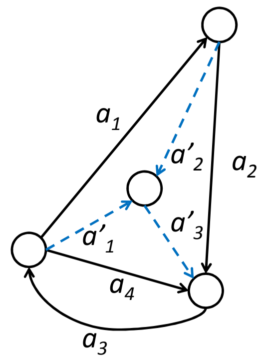

Figure 1: Network matrix is defined by a directed tree (dotted edges) and a directed graph on the same vertex set.

Example 2: Directed Graphs. Let $D = (V, A)$ be a directed graph. Let M be an $|E| \times |V|$ arc-vertex adjacency matrix defined as

$$M_{a,v} = \begin{cases} 0, \text{ if } a \text{ is not incident to } v \\ +1, \text{ if } a \text{ enters } v \\ -1, \text{ if } a \text{ leaves } v \end{cases} \tag{2}$$

M is TUM. This was first observed by Poincaré [1900].

Example 3: Consecutive 1's: A is a consecutive 1's matrix if it is a matrix with entries in $\{0, 1\}$ such that in each row the 1's are in a consecutive block. This naturally arises as an incidence matrix of a collection of intervals and a set of points on the real line.

The above three claims of matrices are special cases of network matrices (due to Tutte).

Definition 14 *A network matrix is defined from a directed graph $D = (V, A)$ and a directed tree $T = (V, A')$ on the same vertex set V. The matrix M is $|A'| \times |A|$ matrix such that for $a = (u, v) \in A$ and $a' \in A'$*

$$M_{a,a'} = \begin{cases} 0, \textit{ if the unique path from } u \to v \textit{ in } T \textit{ does not contain } a' \\ +1, \textit{ if the unique path from } u \to v \textit{ in } T \textit{ passes through } a' \textit{ in forward direction} \\ -1, \textit{ if the unique pathu} \to v \textit{ in } T \textit{ passes through } a' \textit{ in backward direction} \end{cases}$$

The network matrix corresponding to the directed graph and the tree in Figure 1 is given below. The dotted edge is T, and the solid edge is D.

$$M = \begin{array}{c} \\ a_1' \\ a_2' \\ a_3' \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

Theorem 15 *(Tutte) Every network matrix is TUM.*

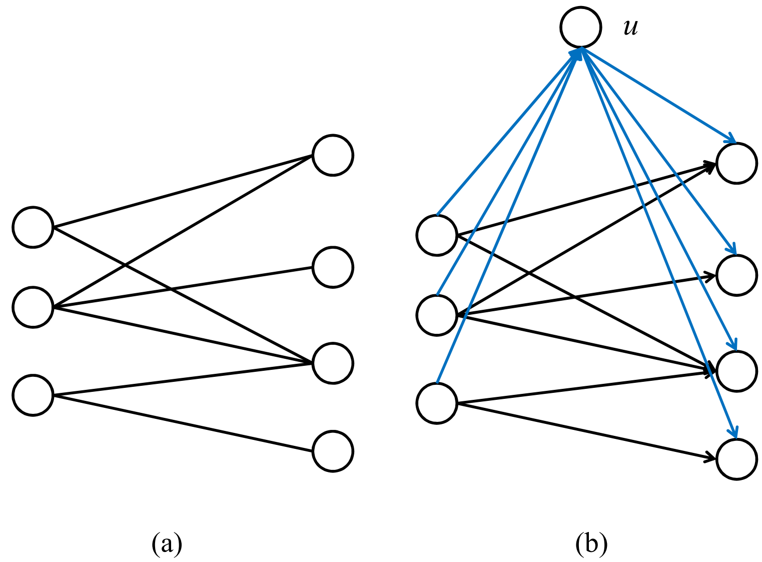

Figure 2:

We will prove this later. First we show that the previous examples can be cast as special cases of network matrices.

Bipartite graphs. Say $G = \{X \cup Y, E\}$ as in Figure 2(a). One can see that edge-vertex adjacency matrix of G as the network matrix induced by a directed graph $G = (X \cup Y \cup \{u\}, A)$ where u is a new vertex and A is the set of arcs defined by orientating the edges of G from X to Y. $T = (X \cup Y \cup \{u\}, A')$ where $A' = \{(v, u)|v \in X\} \cup \{(u, v)|v \in Y\}$ as in Figure 2(b).

Directed graphs. Suppose $D = (V, A)$ is a directed graph. Consider the network matrix induced by $D = (V \cup \{u\}, A)$ and $T = (V \cup \{u\}, A')$ where u is a new vertex and where $A' = \{(v, u)|v \in V\}$.

Consecutive 1's matrix. Let A be a consecutive 1's matrix with m rows and n columns. Assume for simplicity that each row has at least one 1 and let ℓ_i and r_i be the left most and right most columns of the consecutive block of 1's in row i. Let $V = \{1, 2, \ldots, n\}$. Consider $T = (V, A')$ where $A' = \{(i, i + 1) \mid 1 \leq i < n\}$ and $D = (V, A)$ where $A = \{(\ell_i, r_i) \mid 1 \leq i \leq n\}$. It is easy to see that A is the network matrix defined by T and A.

Now we prove that every network matrix is TUM. We need a preliminary lemma.

Lemma 16 *Every submatrix M' of a network matrix M is also a network matrix.*

Proof: If M is a network matrix, defined by $D = (V, A)$ and $T = (V, A')$, then removing a column in M corresponds to removing an arc $a \in A$. Removing a row corresponds to identifying/contracting the end points of an arc a' in T. □

Proposition 17 *A is TUM $\iff$ A' obtained by multiplying any row or column by -1 is TUM.*

Corollary 18 *If M is a network matrix, M is TUM $\iff$ M' is TUM where M' is obtained by reversing an arc of either T or D.*

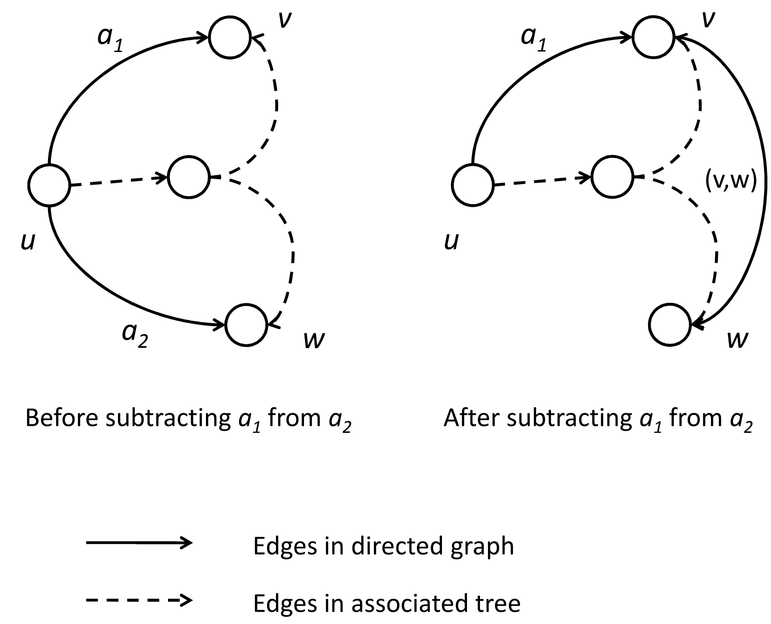

Figure 3:

Proof of Theorem 15. By Lemma 16, it suffices to show that any square network matrix C has determinant in $\{0, +1, -1\}$. Let C be a $k \times k$ network matrix defined by $D = (V, A)$ and $T = (V, A')$. We prove by induction on k that $det(C) \in \{0, 1, -1\}$. Base case with $k = 1$ is trivial since entries of C are in $\{0, 1, -1\}$.

Let $a' \in A'$ be an arc incident to a leaf u in T. By reorienting the arcs of T, we will assume that a' leaves u and moreover all arcs A incident to u leave u (see Corollary 18).

Let $a_1, a_2, \cdots, a_h$ be arcs in A leaving u (If no arcs are incident to u then $det(C) = 0$). Assume without loss of generality that a' is the first row of C and that $a_1, a_2, \cdots, a_h$ are the first h columns of C.

Claim 19 *Let C' be obtained by subtracting column a_1 from column a_2. C' is the network matrix for $T = (V, A')$ and $D = (V, A - a_2 + (v, w))$ where $a_1 = (u, v)$ and $a_2 = (u, w)$.*

We leave the proof of the above as an exercise — see Figure 3.

Let C'' be the matrix obtained by subtracting column of a_1 from each of $a_2, \cdots, a_h$. From the above claim, it is also a network matrix. Moreover, $det(C'') = det(C)$ since determinant is preserved by these operations. Now C'' has 1 in the first row in column one (corresponding to a_1) and 0's in all other columns. Therefore, $det(C'') \in \{0, +1, -1\}$ by expanding along the first row and using induction for the submatrix of C'' consisting of columns 2 to k and rows 2 to k. □

Some natural questions on TUM matrices are the following.

(i) Are there TUM matrices that are not a network matrix (or its transpose)?

(ii) Given a matrix A, can one check efficiently whether it is a TUM matrix?

The answer to (i) is negative as shown by the following two matrices given by Hoffman[1960]

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

and Bixby[1977].

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Amazingly, in some sense, these are the only two exceptions.

Seymour, in a deep and difficult technical theorem, showed via matroid theory methods that any TUM matrix can be obtained by "gluing" together network matrices and the above two matrices via some standard operations that preserve total unimodularity. His descomposition theorem also led to a polynomial time algorithm for checking if a given matrix is TUM. There was an earlier polynomial time algorithm to check if a given matrix is a network matrix. See [1] (Chapters 20 and 21) for details.

References

[1] A. Schrijver. *Theory of Linear and Integer Programming (Paperback).* Wiley, 1998.