CS 598CSC: Combinatorial Optimization	Lecture date: 26 January, 2010
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1 More Background on Polyhedra

This material is mostly from [3].

1.1 Implicit Equalities and Redundant Constraints

Throughout this lecture we will use **affhull** to denote the affine hull, **linspace** to be the linear space, **charcone** to denote the characteristic cone and **convexhull** to be the convex hull. Recall that $P = \{x \mid Ax \leq b\}$ is a *polyhedron* in $\mathbb{R}^n$ where A is a $m \times n$ matrix and b is a $m \times 1$ matrix. An inequality $a_i x \leq b_i$ in $Ax \leq b$ is an *implicit equality* if $a_i x = b_i \ \forall x \in P$. Let $I \subseteq \{1, 2, \dots, m\}$ be the index set of all implicit equalities in $Ax \leq b$. Then we can partition A into $A^= x \leq b^=$ and $A^+ x \leq b^+$. Here $A^=$ consists of the rows of A with indices in I and A^+ are the remaining rows of A. Therefore, $P = \{x \mid A^= x = b^=, A^+ x \leq b^+\}$. In other words, P lies in an affine subspace defined by $A^= x = b^=$.

Exercise 1 *Prove that there is a point $x' \in P$ such that $A^= x' = b^=$ and $A^+ x' < b^+$.*

Definition 1 *The dimension,* $\mathbf{dim}(P)$*, of a polyhedron P is the maximum number of affinely independent points in P minus 1.*

Notice that by definition of dimension, if $P \subseteq \mathbb{R}^n$ then $\mathbf{dim}(P) \leq n$, if $P = \emptyset$ then $\mathbf{dim}(P) = -1$, and $\mathbf{dim}(P) = 0$ if and only if P consists of a single point. If $\mathbf{dim}(P) = n$ then we say that P is *full-dimensional*.

Exercise 2 *Show that* $\mathbf{dim}(P) = n - \mathbf{rank}(A^=)$.

The previous exercise implies that P is full-dimensional if and only if there are no implicit inequalities in $Ax \leq b$.

Definition 2 $\mathbf{affhull}(P) = \{x \mid A^= x = b^=\}$

Definition 3 $\mathbf{linspace}(P) = \{x \mid Ax = 0\} = \mathbf{charcone}(P) \cap -\mathbf{charcone}(P)$. *In other words,* $\mathbf{linspace}(P)$ *is the set of all directions c such that there is a line parallel to c fully contained in P.*

Definition 4 *A polyhedron P is* pointed *if and only if* $\mathbf{linspace}(P) = \{0\}$*, that is* $\mathbf{linspace}(P)$ *has dimension* 0.

A constraint row in $Ax \leq b$ is *redundant* if removing it does not change the polyhedron. The system $Ax \leq b$ is *irredundant* if no constraint is redundant.

1.2 Faces of Polyhedra

Definition 5 *An inequality $\alpha x \leq \beta$, where $\alpha \neq 0$, is a* valid *inequality for a polyhedron $P = \{x \mid Ax \leq b\}$ if $\alpha x \leq \beta\ \forall x \in P$. The inequality is a* supporting hyperplane *if it is valid and has a non-empty intersection with P*

Definition 6 *A* face *of a polyhedron P is the intersection of P with $\{x \mid \alpha x = \beta\}$ where $\alpha x \leq \beta$ is a valid inequality for P.*

We are interested in non-empty faces. Notice that a face of a polyhedron is also a polyhedron. A face of P is an *extreme point* or a *vertex* if it has dimension 0. It is a *facet* if the dimension of the face is $\mathbf{dim}(P) - 1$. The face is an *edge* if it has dimension 1.

Another way to define a face is to say that F is a face of P if $F = \{x \in P \mid A'x = b'\}$ where $A'x \leq b'$ is a subset of the inequalities of $Ax \leq b$. In other words, $F = \{x \in P \mid a_i x = b_i, i \in I\}$ where $I \subseteq \{1, 2, \ldots, m\}$ is a subset of the rows of A.

Now we will show that these two definitions are equivalent.

Theorem 7 *Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^n$. Let $P = \{x \mid Ax \leq b\}$ be a polyhedron. Let F be a face defined by a valid inequality $\alpha x \leq \beta$. Then $\exists I \subseteq \{1.2.\ldots, m\}$ such that $F = \{x \in P \mid a_i x = b_i, i \in I\}$.*

Proof: Let $F = \{x \mid x \in P, \alpha x = \beta\}$ where $\alpha x \leq \beta$ is a supporting hyperplane. Then, the following claim is easy to see.

Claim 8 *F is the set of all optimal solutions to the LP*

$$\begin{aligned} &\max \alpha x \\ &Ax \leq b. \end{aligned}$$

The above LP has an optimal value β. This implies that the dual LP is feasible and has an optimum solution y^*. Let $I = \{i \mid y_i^* > 0\}$. Let X be the set of all optimal solutions to the primal. For any $x' \in X$, by complimentary slackness for x' and y^*, we have that $y_i^* > 0$ implies $a_i x' = b_i$. Therefore X is a subset of the solutions to the following system of inequalities:

$$\begin{aligned} a_i x &= b_i \qquad i \in I \\ a_i x &\leq b_i \qquad i \notin I \end{aligned}$$

Again, by complementary slackness any x' that satisfies the above is optimal (via y^*) for the primal LP and. Therefore $F = X = \{x \in P \mid a_i x_i = b_i, i \in I\}$. $\square$

Now we consider the converse.

Theorem 9 *Let $P = \{x \mid Ax \leq b\}$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Let $I \subseteq \{1, \ldots, m\}$ and $F = \{x \in P \mid a_i x = b_i, i \in I\}$. If F is non-empty, then there is a valid inequality $\alpha x \leq \beta$ such that $F = P \cap \{x \mid \alpha x = \beta\}$.*

Proof:[Idea] Let $\alpha = \sum_{i\in I} a_i$ be a row vector and $\beta = \max\{\alpha x \mid Ax \leq b\}$. We claim that $F = \{x \mid x \in P, \alpha x = \beta\}$ which implies that F is the intersection of P with the supporting hyperplane $\alpha x \leq \beta$. $\square$

Corollary 10

1. *The number of faces of $P = \{x \mid Ax \leq b\}$ where A is a $m \times n$ matrix is at most 2^m.*

2. *Each face is a polyhedron.*

3. *If F is a face of P and $F' \subseteq F$ then F' is a face of P if and only if F' is a face of F.*

4. *The intersecton of two faces is either a face or is empty.*

1.3 Facets

Definition 11 *A* facet *of P is an inclusion-wise maximal face distinct from P. Equivalently, a face F of P is a facet if and only if* $\mathbf{dim}(F) = \mathbf{dim}(P) - 1$.

We have the following theorem about facets.

Theorem 12 *Let $P = \{x \mid Ax \leq b\} = \{x \mid A^{=}x = b^{=}, A^{+}x \leq b^{+}\}$. If no inequality of $A^{+}x \leq b^{+}$ is redundant in $Ax \leq b$, then there is a one to one correspondence between the facets of P and the inequalities in $A^{+}x \leq b^{+}$. That is, F is a facet of P if and only if $F = \{x \in P \mid a_i x = b_i\}$ for some inequality $a_i x \leq b_i$ from $A^{+}x \leq b^{+}$.*

Proof: Let F be a facet of P. Then $F = \{x \in P \mid A'x = b'\}$ where $A'x \leq b'$ is a subsystem of $A^{+}x \leq b^{+}$. Take some inequality $\alpha x \leq \beta$ in $A'x \leq b'$. Then $F' = \{x \in P \mid \alpha x = \beta\}$ is a face of P and $F \subseteq F'$. Moreover, $F' \neq P$ since no inequality in $A^{+} \leq b^{+}$ is an implicit equality.

Let $F = \{x \in P \mid \alpha x = \beta\}$ for some inequality $\alpha x \leq \beta$ from $A^{+} \leq b^{+}$. We claim that $\mathbf{dim}(F) = \mathbf{dim}(P) - 1$ which implies that F is a facet. To prove the claim it is sufficient to show that there is a point $x_0 \in P$ such that $A^{=}x_0 = b^{=}, \alpha x_0 = \beta$ and $A'x_0 < b'$ where $A' \leq b'$ is the inequalities in $A^{+}x \leq b^{+}$ with $\alpha x \leq \beta$ omitted. From Exercise 1, there is a point x_1 such that $\alpha x_1 = \beta$ and $A^{=}x_1 = b^{=}$ and $A^{+}x_1 < b^{+}$. Moreover since $\alpha x \leq \beta$ is irredundat in $Ax \leq b$, there is a point x_2 such that $A^{=}x_2 = b^{=}$ and $A'x_2 \leq b'$ and $\alpha x_2 > \beta$. A convex combination of x_1 and x_2 implies the existence of the desired x_0. $\square$

Corollary 13 *Each face of P is the intersection of some of the facets of P.*

Corollary 14 *A polyhedrom P has no facet if and only if P is an affine subspace.*

Exercise 3 *Prove the above two corollaries using Theorem 12.*

1.4 Minimal Faces and Vertices

A face is inclusion-wise minimal if it does not contain any other face. From Corollary 14 and the fact that a face of a polyhedron is a polyhedron the next proposition follows.

Proposition 15 *A face F of P is minimal if and only if F is an affine subspace.*

Theorem 16 *A set F is minimal face of P if and only if $\emptyset \neq F$, $F \subseteq P$ and $F = \{x \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

Proof: Suppose F is a face and $F = \{x \mid A'x = b'\}$ then by Proposition 15, it is minimal. For the converse direction suppose F is a minimal face of P. Since F is a face, $F = \{x \mid A''x \leq b'', A'x = b'\}$ where $A''x \leq b''$ and $A'x \leq b'$ are two subsystems of $Ax \leq b$. We can assume that $A''x \leq b''$ is as small as possible and therefore, irredundant. From Theorem 12, if $A''x \leq b''$ has any inequality then F has a facet which implies that F is not minimal. Therefore, $F = \{x \mid A'x = b'\}$. $\square$

Exercise 4 *Prove that all minimal faces of a polyhedron $\{x \mid Ax \leq b\}$ are of the form $\{x \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$ where $rank(A') = rank(A)$. Conclude that all minimal faces are translates of* **linspace**(P) *and have the same dimension.*

A *vertex* or an *extreme point* of P is a (minimal) face of dimension 0. That is, a single point. A polyhedron is *pointed* if and only if it has a vertex. Note that since all minimal faces have the same dimension, if P has a vertex than all minimal faces are vertices. Since a minimal face F of P is defined by $A'x = b'$ for some subsystem $A'x \leq b'$ of $Ax \leq b$, if a vertex of P is the *unique solution* to $A'x = b'$ then $\mathbf{rank}(A') = n$. We can then assume that A' has n rows. Vertices are also called *basic feasible solutions.*

Corollary 17 *A polyhedron $\{x \mid Ax \leq b\}$ has a vertex only if A has rank n. The polyhedron $\{x \mid Ax \leq b\}$ is pointed if it is not empty.*

1.5 Decomposition of Polyhedra

Recall that we had earlier stated that,

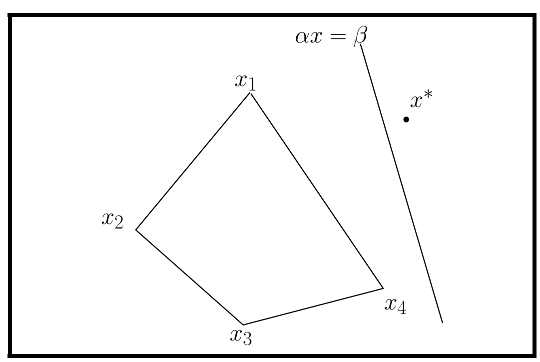
Theorem 18 *Any polyhedron P can be written as $Q + C$ where Q is a convex hull of a finites set of vectors and $C = \{x \mid Ax \leq 0\}$ is the* **charcone** *of P.*

We can give more details of the decomposition now. Given P, let $F_1, F_2, \ldots, F_h$ be its minimal faces. Choose $x_i \in F_i$ arbitrarily. Then $P = \mathbf{convexhull}(x_1, x_2, \ldots, x_h) + C$. In particular, if P is pointed then $x_1, x_2, \ldots, x_h$ are vertices of P and hence $P = \mathbf{convexhull}(\text{vertices}(P)) + C$.

We will prove the above for polytopes.

Theorem 19 *A polytope (bounded polyhedron) is the convex hull of its vertices*

Proof: First observe that a bounded polyhedron is necessarily pointed. Let $X = \{x_1, x_2, \ldots, x_h\}$ be the vertices of P. Clearly $\mathbf{convexhull}(X) \subseteq P$. We prove the converse. Suppose $x^* \in P$ does no belong to $\mathbf{convexhull}(X)$.



Claim 20 *There exists a hyperplane $\alpha x = \beta$ such that $\alpha x_i < \beta \; \forall x_i \in X$ and $\alpha x^* \geq \beta$.*

Proof:[Claim] One can prove this by using Farkas lemma (see [2] for example) or appeal to the general theorem that if two convex sets do not intersect then there is a separating hyperplane for them; in particular if one of the sets is bounded then there is a *strict* separating hyperplane (see [1], Section 2.4). A sketch of this for the restricted case we have is as follows. Let $y \in \mathbf{convexhull}(X)$ minimize the distance from x^* to $\mathbf{convexhull}(X)$. The claim is that a hyperplane that is normal to the line segment joining x^* and y and passing through x^* is the desired hyperplane. Otherwise $\mathbf{convexhull}(X)$ intersects this hyperplane, and let y' be a point in the intersection. Since $\mathbf{convexhull}(X)$ is convex, the line segment joining y' and y is contained in $\mathbf{convexhull}(X)$. Now consider the right angled triangle formed by y, x^*, y'. From elementary geometry, it follows that there is a point closer to x^* than y on the line segment joining y and y', contradicting the choice of y. □

Now consider $\max\{\alpha x \mid x \in P\}$. The set of optimal solutions to this LP is a face of F. By Claim 20, $X \cap F = \emptyset$. Since F is a face of P, it has a vertex of P since P is pointed. This contradicts that X is the set of all vertices of P. □

One consequence of the decomposition theorem is the following.

Theorem 21 *If $P = \{x \mid Ax \leq b\}$ is pointed then for any $c \neq 0$ the LP*

$$\max cx$$
$$Ax \leq b$$

is either unbounded, or there is a vertex x^* such that x^* is an optimal solution.

The proof of the previous theorem is left as an exercise.

2 Complexity of Linear Programming

Recall that LP is an optimization problem of the following form.

$$\max \alpha x$$
$$Ax \leq b$$

As a computational problem we assume that the inputs c, A, b are rational. Thus the input consists of $n + m \times n + n$ rational numbers. Given an instance I we use $\mathbf{size}(I)$ to denote the number of bits in the binary representation of I. We use it loosely for other quantities such as numbers, matrices, etc. We have that $\mathbf{size}(I)$ for an LP instance is,

$$\mathbf{size}(c) + \mathbf{size}(A) + \mathbf{size}(b) \leq (m \times n + 2n)\mathbf{size}(L)$$

where L is the largest number in c, A, b.

Lemma 22 *Given a $n \times n$ rational matrix* $\mathbf{size}(\det(A)) = \mathbf{poly}(\mathbf{size}(A))$.

Proof: For simplicity assume that A has integer entries, otherwise one can multiply each entry by the lcm of the denominators of the rational entries. We have

$$\det(A) = \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

where S_n is the set of all permutations on $\{1, \ldots, n\}$. Hence

$$\begin{aligned} |\det(A)| &\leq \sum_{\sigma \in S_n} \prod_{i=1}^{n} |A_{i,\sigma(i)}| \\ &\leq n! \times L^n \end{aligned}$$

where $L = \max |A_{i,j}|$; here $|x|$ for a number x is its absolute value. Therefore the number of bits required to represent $\det(A)$ is $O(n \log L + n \log n)$ which is polynomial in n and $\log L$, and hence $\mathbf{poly}(\mathbf{size}(A))$. □

Corollary 23 *If A has an inverse, then $\mathbf{size}(A^{-1}) = \mathbf{poly}(\mathbf{size}(A))$.*

Corollary 24 *If $Ax = b$ has a feasible solution then there exists a solution x^* such that $\mathbf{size}(x^*) = \mathbf{poly}(\mathbf{size}(A, b))$.*

Proof: Suppose $Ax = b$ has a feasible solution. By basic linear algbegra, there is a square submatrix U of A with full rank and a sub-vector b' such that $U^{-1}b'$ padded by 0's for the other variables is a feasible solution for the original system. We then apply the previous corollary to U and b'. □

Gaussian elimination can be adapted using the above to show the following — see [3].

Theorem 25 *There is a polynomial time algorith, that given a linear system $Ax = b$, either correctly outputs that it has no feasible solution or outputs a feasible solution. Moreover, the algorithm determines whether A has a unique feasible solution.*

Now we consider the case when $Ax \leq b$ has a feasible solution.

Theorem 26 *If a linear system $Ax \leq b$ has a feasible solution then there exists a solution x^* such that $\mathbf{size}(x^*) = \mathbf{poly}(\mathbf{size}(A, b))$.*

Proof: Consider a minimal face F of $P = \{x \mid Ax \leq b\}$. We have seen that $F = \{x \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$. By Theorem 25, $A'x = b'$ has a solution of size $\mathbf{poly}(\mathbf{size}(A', b'))$. □

Corollary 27 *The problem of deciding whether $\{x \mid Ax \leq b\}$ is non-empty is in* **NP**.

Corollary 28 *The problem of deciding whether $\{x \mid Ax \leq b\}$ is* empty *is in* **NP**. *Equivalently the problem of deciding non-emptiness is in* **coNP**.

Proof: By Farkas' lemma, if $Ax \leq b$ is empty only if $\exists y \geq 0$ such that $yA = 0$ and $yb = -1$. Therefore there is a certificate y the problem of deciding whether $\{x \mid Ax \leq b\}$ is empty. Further, by Lemma 22 this certificate has polynomial size. □

Thus we have seen that deciding whether $Ax \leq b$ is non-empty is in $\mathbf{NP} \cap \mathbf{coNP}$.

Now consider the optimization problem.

$$\max \alpha x$$
$$Ax \leq b$$

A natural decision problem associated with the above problem is to decide if the optimum value is at least some given rational number α.

Exercise 5 *Prove that the above decision problem is in* $\mathbf{NP} \cap \mathbf{coNP}$.

Another useful fact is the following.

Theorem 29 *If the optimum value of the LP* $\max cx$ *such that* $Ax \leq b$ *is finite then the optimum value has size polynomial in the input size.*

Proof:[sketch] If the optimum value is finite then by strong duality then it is achieved by a solution (x', y') that satisfies the following system:

$$cx = yb$$

$$Ax \leq b$$

$$yA = c$$

$$y \geq 0.$$

From Theorem 26, there is a solution (x^*, y^*) to the above system with $\mathbf{size}(x^*, y^*)$ polynomial in $\mathbf{size}(A, b, c)$. Hence the optimum value which is cx^* has size polynomial in $\mathbf{size}(A, b, c)$. □

Exercise 6 *Show that the decision problem of deciding whether* $\max cx$ *where* $Ax \leq b$ *is unbounded is in* $\mathbf{NP} \cap \mathbf{coNP}$.

The optimization problem for

$$\max \alpha x$$
$$Ax \leq b$$

requires an algorithm that correctly outputs one of the following

1. $Ax \leq b$ is infeasible
2. the optimal value is unbounded
3. a solution x^* such that cx^* is the optimum value

A related search problem is given $Ax \leq b$ either output that $Ax \leq b$ is infeasible or a solution x^* such that $Ax^* \leq b$.

Exercise 7 *Prove that the above two search problems are polynomial time equivalent.*

3 Polynomial-time Algorithms for LP

Khachiyan's ellipsoid algorithm in 1978 was the first polynomial-time algorithm for LP. Although an impractical algorithm, it had (and contiues to have) a major theoretical impact. The algorithm shows that one does not need the full system $Ax \leq b$ in advance. If one examines carefully the size of a proof of feasibility of a system of inequalities $Ax \leq b$, one notices that there is a solution x^* such that x^* is a solution to $A'x \leq b'$ for some subsystem $A'x \leq b'$ where rank of A is at most n. This implies that A' can be chosen to have at most n rows. Therefore, if the system has a solution then there is one whose size is polynomial in n and the size of the largest entry in A. We may discuss more details of the ellipsoid method in a later lecture.

Subsequently, Karmarkar in 1984 gave another polynomial-time algorithm using an interior point method. This is much more useful in practice, especially for certain large linear programs, and can beat the simplex method which is the dominant method in practice although it is not a polynomial time algorithm in the worst case.

References

[1] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004. Available electronically at `http://www.stanford.edu/~boyd/cvxbook/`.

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[3] A. Schrijver. *Theory of Linear and Integer Programming (Paperback)*. Wiley, Chapters 7, 8, 1998.