CS 598CSC: Combinatorial Optimization | Lecture data: 4/8/2010
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1 Introduction to Submodular Set Functions and Polymatroids

Submodularity plays an important role in combinatorial optimization. Given a finite ground set S, a *set function* $f : 2^S \to \mathbb{R}$ is *submodular* if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \quad \forall A, B \subseteq S;$$

or equivalently,

$$f(A + e) - f(A) \geq f(B + e) - f(B) \quad \forall A \subseteq B \text{ and } e \in S \setminus B.$$

Another equivalent definition is that

$$f(A + e_1) + f(A + e_2) \geq f(A) + f(A + e_1 + e_2) \quad \forall A \subseteq S \text{ and distinct } e_1, e_2 \in S \setminus A.$$

Exercise: Prove the equivalence of the above three definitions.

A set function $f : 2^S \to \mathbb{R}$ is *non-negative* if $f(A) \geq 0$ $\forall A \subseteq S$. f is *symmetric* if $f(A) = f(S \setminus A)$ $\forall A \subseteq S$. f is *monotone* (*non-decreasing*) if $f(A) \leq f(B)$ $\forall A \subseteq B$. f is *integer-valued* if $f(A) \in \mathbb{Z}$ $\forall A \subseteq S$.

1.1 Examples of submodular functions

Cut functions. Given an undirected graph $G = (V, E)$ and a 'capacity' function $c : E \to \mathbb{R}_+$ on edges, the *cut function* $f : 2^V \to \mathbb{R}_+$ is defined as $f(U) = c(\delta(U))$, *i.e.*, the sum of capacities of edges between U and $V \setminus U$. f is submodular (also non-negative and symmetric, but not monotone).

In an undirected hypergraph $G = (V, \mathcal{E})$ with capacity function $c : \mathcal{E} \to \mathbb{R}_+$, the *cut function* is defined as $f(U) = c(\delta_{\mathcal{E}}(U))$, where $\delta_{\mathcal{E}}(U) = \{e \in \mathcal{E} \mid e \cap U \neq \emptyset \text{ and } e \cap (S \setminus U) \neq \emptyset\}$.

In a directed graph $D = (V, A)$ with capacity function $c : A \to \mathbb{R}_+$, the *cut function* is defined as $f(U) = c(\delta_{\text{out}}(U))$, where $\delta_{\text{out}}(U)$ is the set of arcs leaving U.

Matroids. Let $M = (S, \mathcal{I})$ be a matroid. Then the rank function $r_M : 2^S \to \mathbb{R}_+$ is submodular (also non-negative, integer-valued, and monotone).

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids. Then the function f given by $f(U) = r_{M_1}(U) + r_{M_2}(S \setminus U)$, for $U \subseteq S$, is submodular (also non-negative, and integer-valued). By the matroid intersection theorem, the minimum value of f is equal to the maximum cardinality of a common independent set in the two matroids.

Coverage in set system. Let $T_1, T_2, \ldots, T_n$ be subsets of a finite set T. Let $S = [n] = \{1, 2, \ldots, n\}$ be the ground set. The *coverage function* $f : 2^S \to \mathbb{R}_+$ is defined as $f(A) = |\cup_{i \in A} T_i|$.

A generalization is obtained by introducing the weights $w : T \to \mathbb{R}_+$ of elements in T, and defining the weighted coverage $f(A) = w(\cup_{i \in A} T_i)$.

Another generalization is to introduce a submodular and monotone weight-function $g : 2^T \to \mathbb{R}_+$ of subsets of T. Then the function f is defined as $f(A) = g(\cup_{i \in A} T_i)$.

All the three versions of f here are submodular (also non-negative, and monotone).

Flows to a sink. Let $D = (V, A)$ be a directed graph with an arc-capacity function $c : A \to \mathbb{R}_+$. Let a vertex $t \in V$ be the *sink*. Consider a subset $S \subseteq V \setminus \{t\}$ of vertices. Define a function $f : 2^S \to \mathbb{R}_+$ as $f(U)$ = max flow from U to t in the directed graph D with edge capacities c, for a set of 'sources' U. Then f is submodular (also non-negative and monotone).

Max element. Let S be a finite set and let $w : S \to \mathbb{R}$. Define a function $f : 2^S \to \mathbb{R}$ as $f(U) = \max\{w(u) \mid u \in U\}$ for nonempty $U \subseteq S$, and $f(\emptyset) = \min\{w(u) \mid u \in S\}$. Then f is submodular (also monotone).

Entropy and Mutual information. Let $X_1, X_2, \ldots, X_n$ be random variables over some underlying probability space, and $S = \{1, 2, \ldots, n\}$. For $A \subseteq S$, define $X_A = \{X_i \mid i \in A\}$ to be the set of random variables with indices in A. Then $f(A) = H(X_A)$, where $H(\cdot)$ is the entropy function, is submodular (also non-negative and monotone). Also, $f(A) = I(X_A; X_{S\setminus A})$, where $I(\cdot;\cdot)$ is the mutual information of two random variables, is submodular.

Exercise: Prove the submodularity of the functions introduced in this subsection.

1.2 Polymatroids

Define two polyhedra associated with a set function f on S:

$$P_f = \{x \in \mathbb{R}^S \mid x(U) \leq f(U)\ \forall U \subseteq S,\ x \geq \mathbf{0}\} \text{ and } EP_f = \{x \in \mathbb{R}^S \mid x(U) \leq f(U)\ \forall U \subseteq S\}.$$

If f is a submodular function, then P_f is called the *polymatroid associated with* f, and EP_f the *extended polymatroid associated with* f. A polyhedron is called an (extended) polymatroid if it is the (extended) polymatroid associated with some submodular function. Since $0 \leq x_s \leq f(\{s\})$ for each $s \in S$, a polymatroid is bounded, and hence is a polytope.

An observation is that P_f is non-empty iff $f \geq \mathbf{0}$, and EP_f is non-empty iff $f(\emptyset) \geq 0$.

If f is the rank function of a matroid M, then P_f is the independent set polytope of M.

A vector x in EP_f (or in P_f) is called a *base vector* of EP_f (or of P_f) if $x(S) = f(S)$. A *base vector* of f is a base vector of EP_f. The set of all base vectors of f is called the *base polytope* of EP_f or of f. It is a face of EP_f and denoted by B_f:

$$B_f = \{x \in \mathbb{R}^S \mid x(U) \leq f(U)\ \forall U \subseteq S,\ x(S) = f(S)\}.$$

B_f is a polytope, since $f(\{s\}) \geq x_s = x(S) - x(S \setminus \{s\}) \geq f(S) - f(S \setminus \{s\})$ for each $s \in S$.

The following claim is about the set of tight constraints in the extended polymatroid associated with a submodular function f.

Claim 1 *Let $f : 2^S \to \mathbb{R}$ be a submodular set function. For $x \in EP_f$, define $\mathcal{F}_x = \{U \subseteq S \mid x(U) = f(U)\}$ (tight constraints). Then $\mathcal{F}_x$ is closed under taking unions and intersections.*

Proof: Consider any two sets $U, V \in \mathcal{F}_x$, we have

$$f(U \cup V) \geq x(U \cup V) = x(U) + x(V) - x(U \cap V) \geq f(U) + f(V) - f(U \cap V) \geq f(U \cup V).$$

Therefore, $x(U \cup V) = f(U \cup V)$ and $x(U \cap V) = f(U \cap V)$. □

Given a submodular set function f on S and a vector $a \in \mathbb{R}^S$, define the set function $f|a$ as

$$(f|a)(U) = \min_{T \subseteq U}(f(T) + a(U \setminus T)).$$

Claim 2 *If f is a submodular set function on S, $f|a$ is also submodular.*

Proof: Let $g = f|a$ for the simplicity of notation. For any $X, Y \subseteq S$, let $X' \subseteq X$ s.t. $g(X) = f(X') + a(X \setminus X')$, and $Y' \subseteq Y$ s.t. $g(Y) = f(Y') + a(Y \setminus Y')$. Then, from the definition of g,

$$g(X \cap Y) + g(X \cup Y) \leq \big(f(X' \cap Y') + a((X \cap Y) \setminus (X' \cap Y'))\big) + \big(f(X' \cup Y') + a((X \cup Y) \setminus (X' \cup Y'))\big).$$

From the submodularity of f,

$$f(X' \cap Y') + f(X' \cup Y') \leq f(X') + f(Y').$$

And from the modularity of a,

$$\begin{aligned} a((X \cap Y) \setminus (X' \cap Y')) + a((X \cup Y) \setminus (X' \cup Y')) &= a(X \cap Y) + a(X \cup Y) - a(X' \cap Y') - a(X' \cup Y') \\ &= a(X) + a(Y) - a(X') - a(Y'). \end{aligned}$$

Therefore, we have $g(X \cap Y) + g(X \cup Y) \leq f(X') + f(Y') + a(X \setminus X') + a(Y \setminus Y')$. □

What is $EP_{f|a}$ and $P_{f|a}$? We have the following claim.

Claim 3 *If f is a submodular set function on S and $f(\emptyset) = 0$, $EP_{f|a} = \{x \in EP_f \mid x \leq a\}$ and $P_{f|a} = \{x \in P_f \mid x \leq a\}$.*

Proof: For any $x \in EP_{f|a}$ and any $U \subseteq S$, we have that $x(U) \leq (f|a)(U) \leq f(U) + a(U \setminus U) = f(U)$ implying $x \in EP_f$, and that $x(U) \leq (f|a)(U) \leq f(\emptyset) + a(U \setminus \emptyset) = a(U)$, implying $x \leq a$.

For any $x \in EP_f$ with $x \leq a$ and any $U \subseteq S$, suppose that $(f|a)(U) = f(T) + a(U \setminus T)$. Then we have, $x(U) = x(T) + x(U \setminus T) \leq f(T) + a(U \setminus T) = (f|a)(U)$, implying $x \in EP_{f|a}$.

The proof of $P_{f|a} = \{x \in P_f \mid x \leq a\}$ is similar. □

A special case of the above claim is that when $a = \mathbf{0}$, then $(f|\mathbf{0})(U) = \min_{T \subseteq U} f(T)$ and $EP_{f|\mathbf{0}} = \{x \in EP_f \mid x \leq \mathbf{0}\}$.

2 Optimization over Polymatroids by the Greedy Algorithm

Let $f : 2^S \to \mathbb{R}$ be a submodular function and assume it is given as a value oracle. Also given a weight vector $w : S \to \mathbb{R}_+$, we consider the problem of maximizing $w \cdot x$ over EP_f.

$$\begin{aligned} &\max w \cdot x \\ &x \in EP_f. \end{aligned} \tag{1}$$

Edmonds showed that the greedy algorithm for matroids can be generalized to this setting.

We assume (or require) that $w \geq \mathbf{0}$, because otherwise, the maximum value is unbounded. W.l.o.g., we can assume that $f(\emptyset) = 0$: if $f(\emptyset) < 0$, $EP_f = \emptyset$; and if $f(\emptyset) > 0$, setting $f(\emptyset) = 0$ does not violate the submodularity.

Greedy algorithm and integrality. Consider the following greedy algorithm:

1. Order $S = \{s_1, s_2, \ldots, s_n\}$ s.t. $w(s_1) \geq \ldots \geq w(s_n)$. Let $A_i = \{s_1, \ldots, s_i\}$ for $1 \leq i \leq n$.
2. Define $A_0 = \emptyset$ and let $x'(s_i) = f(A_i) - f(A_{i-1})$, for $1 \leq i \leq n$.

Note that the greedy algorithm is a strongly polynomial-time algorithm.

To show that the greedy algorithm above is correct, consider the dual of maximizing $w \cdot x$:

$$\min \sum_{U \subseteq S} y(U) f(U) \tag{2}$$
$$\sum_{U \ni s_i} y(U) = w(s_i)$$
$$y \geq \mathbf{0}.$$

Define the dual solution: $y'(A_n) = y'(S) = w(s_n)$, $y'(A_i) = w(s_i) - w(s_{i+1})$ for $1 \leq i \leq n-1$, and $y'(U) = 0$ for all other $U \subseteq S$.

Exercise: Prove that x' and y' are feasible and y' satisfies complementary slackness w.r.t. x' in (1) and (2). Then it follows that the system of inequalities $\{x \in \mathbb{R}^S \mid x(U) \leq f(U),\ \forall U \subseteq S\}$ is totally dual integral (TDI), because the optimum of (2) is attained by the integral vector y' constructed above (if the optimum exists and is finite).

Theorem 4 *If $f : 2^S \to \mathbb{R}$ is a submodular function with $f(\emptyset) = 0$, the greedy algorithm (computing x') gives an optimum solution to (1). Moreover, the system of inequalities $\{x \in \mathbb{R}^S \mid x(U) \leq f(U),\ \forall U \subseteq S\}$ is totally dual integral (TDI).*

Now consider the case of P_f. Note that P_f is non-empty iff $f \geq \mathbf{0}$. We note that if f is monotone and non-negative, then the solution x' produced by the greedy algorithm satisfies $x \geq \mathbf{0}$ and hence if feasible for P_f. So we obtains:

Corollary 5 *If f is a non-negative monotone submodular function on S with $f(\emptyset) = 0$ and let $w : S \to \mathbb{R}_+$, then the greedy algorithm also gives an optimum solution x' to $\max\{w \cdot x \mid x \in P_f\}$. Moreover, the system of inequalities $\{x \in \mathbb{R}_+^S \mid x(U) \leq f(U),\ \forall U \subseteq S\}$ is TDI.*

Therefore, from Theorem 4 and Corollary 5, for any integer-valued submodular function f, EP_f is an integer polyhedron, and if in addition f is non-negative and monotone, P_f is also an integer polyhedron.

One-to-one correspondence between f and EP_f. Theorem 4 also implies f can be recovered from EP_f. In other words, for any extended polymatroid P, there is a unique submodular function f satisfying $f(\emptyset) = 0$, with which P is associated with (*i.e.*, $EP_f = P$), since:

Claim 6 *Let f be a submodular function on S with $f(\emptyset) = 0$. Then $f(U) = \max\{x(U) \mid x \in EP_f\}$ for each $U \subseteq S$.*

Proof: Let $\alpha = \max\{x(U) \mid x \in EP_f\}$. $\alpha \leq f(U)$, because $x \in EP_f$. To prove $\alpha \geq f(U)$, in (1), define $w(s_i) = 1$ iff $s_i \in U$ and $w(s_i) = 0$ otherwise, consider the greedy algorithm producing x':

W.l.o.g., we can assume after Step 1 in the greedy algorithm, $U = \{s_1, s_2, \ldots, s_k\}$, and $w(s_i) = 1$ if $1 \leq i \leq k$ and $w(s_i) = 0$ otherwise. Define $x'(s_i) = f(A_i) - f(A_{i-1})$ where $A_i = \{s_1, \ldots, s_i\}$. As x' is feasible in (1) (exercise: $x' \in EP_f$), $w \cdot x' \leq \max\{w \cdot x \mid x \in EP_f\}$. From the definition of w, $w \cdot x = x(U)$, and from the selection of x', $w \cdot x' = f(A_1) - f(\emptyset) + f(A_2) - f(A_1) + \ldots + f(A_k) - f(A_{k-1}) = f(A_k) - f(\emptyset) = f(U)$. Therefore, $f(U) \leq \max\{x(U) \mid x \in EP_f\} = \alpha$. □

There is a similar one-to-one correspondence between non-empty polymatroids and non-negative monotone submodular functions f with $f(\emptyset) = 0$. We can also show that, for any such function f, $f(U) = \max\{x(U) \mid x \in P_f\}$ for each $U \subseteq S$.

3 Ellipsoid-based Submodular Function Minimization

Let $f : 2^S \to \mathbb{R}$ be a submodular function and assume it is given as a value oracle, *i.e.*, when given $U \subseteq S$, the oracle returns $f(U)$. Our goal is to find $\min_{U\subseteq S} f(U)$. Before discussing combinatorial algorithms for this problem, we will first describe an algorithm based on the equivalence of optimization and separation (the ellipsoid-based method) in this section.

We can assume $f(\emptyset) = 0$ (by resetting $f(U) \leftarrow f(U) - f(\emptyset)$ for all $U \subseteq S$). With the greedy algorithm introduced in Section 2, we can optimize over EP_f in polynomial time (Theorem 4). So the separation problem for EP_f is solvable in polynomial time, hence also the separation problem for $P = EP_f \cap \{x \mid x \leq \mathbf{0}\}$, and therefore also the optimization problem for P.

Fact 7 *There is a polynomial-time algorithm to separate over P, and hence to optimize over P.*

Claim 8 *If $f(\emptyset) = 0$, $\max\{x(S) \mid x \in P\} = \min_{U\subseteq S} f(U)$, where $P = EP_f \cap \{x \mid x \leq \mathbf{0}\}$.*

Proof: Define $g = f|\mathbf{0}$, and then we have $g(S) = \min_{U\subseteq S} f(U)$. Since g is submodular (from Claim 2) and $P = EP_g$ (from Claim 3), thus from Claim 6, $g(S) = \max\{x(S) \mid x \in P\}$. Therefore, we have $\max\{x(S) \mid x \in P\} = \min_{U\subseteq S} f(U)$. □

Fact 7 and Claim 8 imply that we can compute the value of $\min_{U\subseteq S} f(U)$ in polynomial time. We still need an algorithm to find $U^* \subseteq S$ s.t. $f(U^*) = \min_{U\subseteq S} f(U)$.

Theorem 9 *There is a polynomial-time algorithm to minimize a submodular function f given by a value oracle.*

Proof: To complete the proof, we present an algorithm to find $U^* \subseteq S$ s.t. $f(U^*) = \min_{U\subseteq S} f(U)$.

Initially, let $\alpha = \min_{U\subseteq S} f(U)$. In each iteration:

1. We find an element $s \in S$ s.t. the minimum value of f over all subsets of $S \setminus \{s\}$ is equal to α, which implies that there exists an $U^* \subseteq S$ with $f(U^*) = \alpha$ and $s \notin U^*$.

2. So we then focus on $S \setminus \{s\}$ for finding the U^*; this algorithm proceeds with setting $S \leftarrow S \setminus \{s\}$ and repeats Step 1 for finding another such s; if such an s cannot be found in some iteration, the algorithm terminates and returns the current S as U^*. □