CS 598CSC: Combinatorial Optimization — **Lecture date: March 18, 2010**
Instructor: Chandra Chekuri — **Scribe: Jason Sauppe**

1 Weighted Matroid Intersection

We saw an algorithm for finding a maximum cardinality set in the intersection of two matroids. The algorithm generalized in a straightforward fashion to the weighted case. The correctness is more complicated and we will not discuss it here.

The algorithm for the weighted case is also an augmenting path algorithm. Recall the cardinality algorithm 1:

Algorithm 1 Algorithm for Maximum Cardinality Independent Set in Intersection of Two Matroids

procedure MAXINDEPSET($M_1 = (S, \mathcal{I}_1)$, $M_2 = (S, \mathcal{I}_2)$)
    $I \leftarrow \emptyset$
    repeat
        Construct $D_{M_1,M_2}(I)$
        $X_1 \leftarrow \{z \in S \backslash I \mid I + z \in \mathcal{I}_1\}$
        $X_2 \leftarrow \{z \in S \backslash I \mid I + z \in \mathcal{I}_2\}$
        Let P be a shortest $X_1 - X_2$ path in $D_{M_1,M_2}(I)$
        if P is not empty **then**
            $I \leftarrow I \Delta V(P)$ ▷ $I' = (I \backslash \{y_1, \ldots, y_t\}) \cup \{z_0, z_1, \ldots, z_t\}$
        end if   ▷ Else P is empty and I is maximal
    until I is maximal
end procedure

The weighted case differs only in finding P. Let $w : S \rightarrow \Re^+$ be the weights. Then in computing P we assign weights to each vertex $x \in D_{M_1,M_2}(I)$ as $w(x)$ if $x \in I$ and $-w(x)$ to $x \notin I$. The desired path P should now be a minimum length path according to the weights; further, P should have the smallest number of arcs among all minimum length paths.

Theorem 1 *There is a polynomial time combinatorial algorithm for weighted matroid intersection.*

2 Matroid Intersection Polytope

Edmonds proved the following theorem about the matroid intersection polytope:

Theorem 2 *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids on S. Then the convex hull of the characteristic vectors of sets in $\mathcal{I}_1 \cap \mathcal{I}_2$ is determined by the following set of inequalities:*

$$\begin{aligned} x &\geq 0 \\ x(U) &\leq r_1(U) \qquad \forall\, U \subseteq S \\ x(U) &\leq r_2(U) \qquad \forall\, U \subseteq S \end{aligned}$$

where r_1 and r_2 are the rank functions of M_1 and M_2, respectively. Moreover, the system of inequalities is TDI. In other words,

$$P_{\textit{common indep. set}}(M_1, M_2) = P_{\textit{indep. set}}(M_1) \cap P_{\textit{indep. set}}(M_2).$$

Proof: Consider the primal-dual pair

$$\begin{aligned}
\max \sum_{e \in S} & w(e)x(e) \\
\text{subject to} \quad & x(U) \leq r_1(U) \qquad \forall\, U \subseteq S \\
& x(U) \leq r_2(U) \qquad \forall\, U \subseteq S \\
& x \geq 0
\end{aligned}$$

$$\begin{aligned}
\min \quad & \sum_{U \subseteq S} \Big(r_1(U) y_1(U) + r_2(U) y_2(U) \Big) \\
\text{subject to} \quad & \sum_{\substack{U \subseteq S \\ U \ni e}} \Big(y_1(U) + y_2(U) \Big) \geq w(e) \qquad \forall\, e \in S \\
& y_1 \geq 0 \\
& y_2 \geq 0
\end{aligned}$$

We will prove that the dual has an integral optimum solution whenever w is integral. We can assume that $w(e) \geq 0$ for each e without loss of generality.

Lemma 3 *There exists an optimum solution y_1^*, y_2^* to the dual such that*

$$\mathcal{F}_1 = \{U \subseteq S \mid y_1^*(U) > 0\}$$
$$\mathcal{F}_2 = \{U \subseteq S \mid y_2^*(U) > 0\}$$

are chains.

Proof: Suppose that no optimum solution to the dual satisfies the above property. Then choose an optimum y_1^*, y_2^* with $\mathcal{F}_1 = \{U \subseteq S \mid y_1^*(U) > 0\}$ and $\mathcal{F}_2 = \{U \subseteq S \mid y_2^*(U) > 0\}$ such that the number of proper intersections plus the number of disjoint sets in $\mathcal{F}_1$ and $\mathcal{F}_2$ is minimal.

Then for $A, B \in \mathcal{F}_1$, if A and B properly intersect or are disjoint, we can increase $y_1^*(A \cap B)$ and $y_1^*(A \cup B)$ by ϵ and decrease $y_1^*(A)$ and $y_1^*(B)$ by ϵ to create a new dual solution. This new solution is still dual feasible since

$$\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B).$$

and the dual objective value changes by

$$-\epsilon \Big(r_1(A) + r_1(B) \Big) + \epsilon \Big(r_1(A \cup B) + r_1(A \cap B) \Big).$$

By the submodularity of r_1, this is ≤ 0. If this value is < 0, then this contradicts the optimality of the original solution y_1^*, y_2^*. On the other hand, if this value equals 0, then we have a new optimum solution for the dual with a smaller number of proper intersections plus disjoint sets in $\mathcal{F}_1, \mathcal{F}_2$, contradicting the choice of y_1^*, y_2^*. This follows similarly for $A, B \in \mathcal{F}_2$. $\square$

Corollary 4 *There exists a vertex solution y_1^*, y_2^* such that the support of y_1^* and y_2^* are chains.*

Proof Sketch. Suppose that no vertex solution to the dual satisfies this property. Then choose a vertex solution y_1^*, y_2^* with $\mathcal{F}_1 = \{U \subseteq S \mid y_1^*(U) > 0\}$ and $\mathcal{F}_2 = \{U \subseteq S \mid y_2^*(U) > 0\}$ such that the number of proper intersections plus the number of disjoint sets in $\mathcal{F}_1$ and $\mathcal{F}_2$ is minimal. Perform the uncrossing technique as in the previous proof.

Then for all $e \in S$, the constraint for e remains tight after uncrossing. This holds trivially for $e \notin A \cup B$. If $e \in A$ but $e \notin B$, then $e \in A \cup B$ but $e \notin A \cap B$, so the net change in the constraint for e is $\epsilon - \epsilon = 0$ and it therefore remains at equality (similarly for $e \in B$ but $e \notin A$). If $e \in A \cap B$, then the net change in the constraint for e is $2\epsilon - 2\epsilon = 0$ and the contraint remains tight.

The uncrossing technique then creates a new vertex solution with fewer proper intersections plus disjoint sets in $\mathcal{F}_1$ and $\mathcal{F}_2$, contradicting the choice of y_1^*, y_2^*. $\square$

The following very useful lemma was shown by Edmonds.

Lemma 5 *Let S be a set and $\mathcal{F}_1$ and $\mathcal{F}_2$ be two laminar families on S. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and let A be the $S \times \mathcal{F}$ incidence matrix. Then A is TUM.*

Proof: By $S \times \mathcal{F}$ incidence matrix we mean $A_{x,C} = 1$ if $x \in C$ where $x \in S$, $C \in \mathcal{F}$, and $A_{x,C} = 0$ otherwise. Without loss of generality, we can assume that each $x \in S$ appears in at least one set C in either $\mathcal{F}_1$ or $\mathcal{F}_2$, otherwise we can remove x from S without affecting A (since the row for x would consist of all 0's).

Let A be a counterexample with $|\mathcal{F}| + |S|$ minimal, and among such with a minimal number of 1's in A. There are two cases to consider: First, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are collections of disjoint sets, then every row of A has at most two nonzero entries. If every row of A has exactly two nonzero entries, then A represents the edge-vertex incidence matrix of a bipartite graph, which is TUM (see Figure 1). Since A is a counterexample, this cannot be the case. Therefore, at least one row of A must have only one nonzero entry.

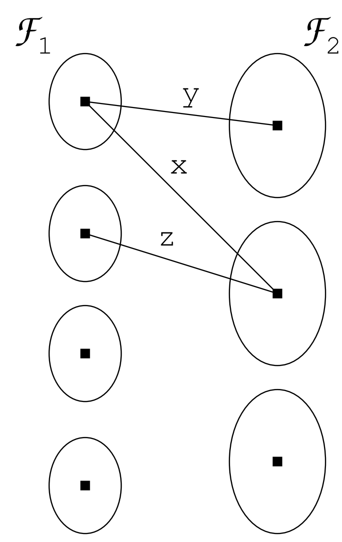

Figure 1: Case where $\mathcal{F}_1$ and $\mathcal{F}_2$ are collections of disjoint sets

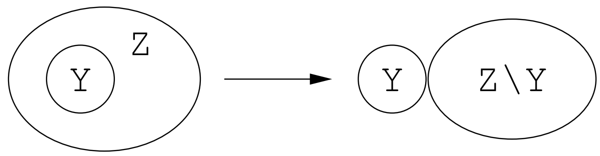

Figure 2: Replacing $Y, Z \in \mathcal{F}_1$ with $Y, Z\backslash Y$

Let A' be the the matrix consisting of the rows of A with two nonzero entries. Then A' is TUM (for the reason given before). We claim that this implies that A is also TUM: for any square submatrix B of A, the determinant of B can be computed by first expanding the computation along the rows of B with only one nonzero entry. The resulting submatrix represents a square submatrix of A', which does not have determinant -2 or 2. Therefore, the original submatrix B of A cannot have determinant -2 or 2. This is true for any square submatrix of A, so A must be TUM.

This means that $\mathcal{F}_1$ and $\mathcal{F}_2$ cannot be collections of disjoint sets if A is to be a counterexample. Therefore, at least one of $\mathcal{F}_1$ and $\mathcal{F}_2$ must have two sets that are not disjoint. Without loss of generality, assume that $\mathcal{F}_1$ has at least two sets Z and Y such that $Y \subset Z$. Of all possible Z and Y meeting this criteria, choose the smallest Z. Then replacing Z by $Z\backslash Y$ generates another laminar family $\mathcal{F}_1'$.

Let A' be the incidence matrix for S and $\mathcal{F}' = \mathcal{F}_1' \cup \mathcal{F}_2$.

Claim 6 *A' is TUM if and only if A is TUM.*

A' is obtained from A by subtracting the column for Y from the column for Z. Hence the determinants of all submatrices are preserved. Also, A' has fewer 1's than A since $Y \neq \emptyset$. Since A was chosen as a counterexample with the smallest number of 1's, A' cannot be a counterexample, and thus A' is TUM. But this implies that A is TUM, as well. Therefore, no such counterexample A can exist. $\square$

Let y_1^*, y_2^* be a vertex solution such that $\mathcal{F}_1$ and $\mathcal{F}_2$ are chains. Since y_1^*, y_2^* is a vertex solution, we have a subset $\mathcal{F} \subseteq \mathcal{F}_1 \cup \mathcal{F}_2$ such that (y_1^*, y_2^*) is a solution to the system of equalities

$$\sum_{\substack{U \in \mathcal{F} \\ U \ni e}} \Big(y_1(U) + y_2(U) \Big) = w(e) \qquad \forall\ e \in S.$$

Then by Lemma 5, the constraint matrix for the above system corresponds to a TUM matrix. This implies that there is an integral solution y_1, y_2 for integral w. From this we can conclude that the dual LP has an integral optimum solution whenever w is integral, and therefore the system of inequalities for the matroid intersection polytope is TDI. $\square$