1 Matroid Intersection

One of several major contributions of Edmonds to combinatorial optimization is algorithms and polyhedral theorems for matroid intersection, and more generally polymatroid intersection.

From an optimization point of view, the matroid intersection problem is the following: Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids on the same ground set S. Then $\mathcal{I}_1 \cap \mathcal{I}_2$ is the collection of all sets that are independent in both matroids.

One can ask the following algorithmic questions:

1. Is there a common base in the two matroids? That is, is there $\mathcal{I} \in \mathcal{B}_1 \cap \mathcal{B}_2$ where $\mathcal{B}_1$ and $\mathcal{B}_2$ are the bases of M_1 and M_2.
2. Output a maximum cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$.
3. Given $w : S \rightarrow \Re$, output a maximum weight set in $\mathcal{I}_1 \cap \mathcal{I}_2$. Or output a maximum weight common base, if it exists.

Remark 1 *It is easy to see that the intersection of two matroids, i.e., $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$, is not necessarily a matroid.*

Exercise 2 *If $M_1 = (S, \mathcal{I}_1)$ is a matroid and $M_2 = (S, \mathcal{I}_2)$ is the uniform matroid, then $M_3 = (S, \mathcal{I}_1 \cap \mathcal{I}_2)$ is a matroid.*

As one can imagine, matroid intersection can capture several additional optimization problems.

Example: Bipartite Matching. Let $G = (V, E)$ be a bipartite graph with bipartition $A \cup B$. Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two partition matroids on E, where

$$\mathcal{I}_1 = \{E' \subseteq E \mid |\delta(v) \cap E'| \leq 1,\ v \in A\}$$
$$\mathcal{I}_2 = \{E' \subseteq E \mid |\delta(v) \cap E'| \leq 1,\ v \in B\}.$$

Then it is easy to see that $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ if and only if I induces a matching in G. Thus bipartite matching problems are special cases of matroid intersection problems.

Example: Branchings and Arborescences. Let $D = (V, A)$ be a directed graph. A *branching* in D is a set of edges $A' \subseteq A$ such that the in-degree of each node is at most one and the edges in A form a forest. (An example is shown in Figure 1.) An *arborescence* rooted at a node $r \in V$ is a directed out-tree such that r has a path to each node $v \in V$. Thus an arborescence is a branching in which r is the only node with in-degree 0.

Consider two matroids $M_1 = (A, \mathcal{I}_1)$ and $M_2 = (A, \mathcal{I}_2)$ where $M_1 = (A, \mathcal{I}_1)$ is a partition matroid:

$$\mathcal{I}_1 = \{A' \subseteq A \mid |\delta^-(v) \cap A'| \leq 1,\ v \in V\}$$

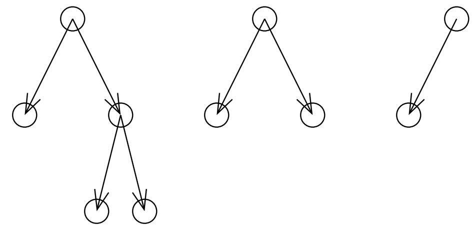

Figure 1: Example of a branching

and M_2 is a graphic matroid on $G = (V, A^u)$ obtained by making an undirected graph on V by removing directions from arcs in A with:

$$\mathcal{I}_2 = \{A' \subseteq A \mid A' \text{ induces a forest in } G^u\}$$

It is easy to see that $\mathcal{I}_1 \cap \mathcal{I}_2$ is the set of all branchings, and a common basis corresponds to arborescences.

Example: Colorful Spanning Trees. Let $G = (V, E)$ where edges in E are colored with k colors. That is, $E = E_1 \uplus E_2 \uplus \ldots \uplus E_k$. Suppose we are given integers $h_1, h_2, \ldots, h_k$ and wish to find a spanning tree that has at most h_i edges of color i (i.e., from E_i). Observe that this can be phrased as a matroid intersection problem: it is the combination of a spanning tree matroid and a partition matroid.

We now state a min-max theorem for the size of the maximum cardinality set in the intersection of two matroids.

Theorem 3 *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids with rank functions r_1 and r_2. Then the size of the maximum cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by:*

$$\min_{U \subseteq S} \; r_1(U) + r_2(S \backslash U)$$

Proof: Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. Take any set $U \subseteq S$. Then

$$I = |I \cap U| + |I \backslash U| \leq r_1(U) + r_2(S \backslash U)$$

since $I \cap U \in \mathcal{I}_1$ and $I \backslash U \in \mathcal{I}_2$. □

We prove the difficult direction algorithmically. That is, we describe an algorithm for the maximum cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$ that, as a byproduct, proves the other direction.

The algorithm is an "augmenting" path type algorithm inspired by bipartite matching and matroid base exchange properties that we discussed earlier. Given $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, the algorithm outputs a $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $|J| = |I| + 1$, or certifies correctly that I is a maximum cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$ by exhibiting a set $U \subseteq S$ such that $|I| = r_1(U) + r_2(S \backslash U)$.

Recall that for a matroid $M = (S, \mathcal{I})$ and $I \in \mathcal{I}$, we defined a directed graph $D_M(I) = (S, A(I))$ where

$$A(I) = \{(y, z) \mid y \in I, \; z \in S \backslash I, \; I - y + z \in \mathcal{I}\}$$

as a graph that captures exchanges for I.

Now we have two matroids M_1 and M_2 and $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ and we wish to augment I to another set $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ if possible. For this purpose we define a graph $D_{M_1,M_2}(I) = (S, A(I))$ where

$$\begin{aligned} A(I) = \{(y,z) \mid y \in S,\ z \in S \backslash I,\ I - y + z \in \mathcal{I}_1\} \\ \cup \{(z', y') \mid z' \in S \backslash I,\ y' \in I,\ I - y' + z' \in \mathcal{I}_2\} \end{aligned}$$

In other words, $D_{M_1,M_2}(I)$ is the union of $D_{M_1}(I)$ and the reverse of $D_{M_2}(I)$. In this sense there is asymmetry in M_1 and M_2. (An example is shown in Figure 2.)

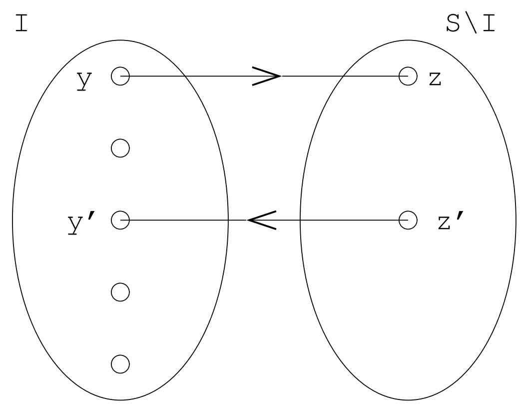

Figure 2: Exchange Graph $D_{M_1,M_2}(I)$

$(y,z) \in A(I) \Rightarrow I - y + z \in \mathcal{I}_1$

$(z',y') \in A(I) \Rightarrow I - y' + z' \in \mathcal{I}_2$

Let $X_1 = \{z \in S \backslash I \mid I + z \in \mathcal{I}_1\}$ and $X_2 = \{z \in S \backslash I \mid I + z \in \mathcal{I}_2\}$, and let P be a <u>shortest</u> path from X_1 to X_2 in $D_{M_1,M_2}(I)$. Note that the shortest path could consist of a single $z \in X_1 \cap X_2$. There may not be any path P between X_1 and X_2.

Lemma 4 *If there is no $X_1 - X_2$ path in $D_{M_1,M_2}(I)$, then I is a maximum cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$.*

Proof: Note that if X_1 or X_2 are empty then I is a base in one of M_1 or M_2 and hence a max cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$. So assume $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. Let U be the set of nodes that can reach X_2 in $D_{M_1,M_2}(I)$. No $X_1 - X_2$ path implies that $X_1 \cap U = \emptyset$, $X_2 \subseteq U$, and $\delta^-(U) = \emptyset$ (i.e., no arcs enter U). Then we have the following:

Claim 5 $r_1(U) \leq |I \cap U|$

Proof: If $r_1(U) > |I \cap U|$, then $\exists z \in U \backslash (I \cap U)$ such that $(I \cap U) + z \in \mathcal{I}_1$ with $I + z \notin \mathcal{I}_1$. If $I + z \in \mathcal{I}_1$, then $z \in X_1$ and $X_1 \cap U \neq \emptyset$, contradicting the fact that there is no $X_1 - X_2$ path. Since $(I \cap U) + z \in \mathcal{I}_1$ but $I + z \notin \mathcal{I}_1$, there must exist a $y \in I \backslash U$ such that $I - y + z \in \mathcal{I}_1$. But then $(y,z) \in A(I)$, contradicting the fact that $\delta^-(U) = \emptyset$ (shown in Figure 3).

□

Claim 6 $r_2(S \backslash U) \leq |I \backslash U|$ *(The proof is similar to the previous proof.)*

Thus $|I| = |I \cap U| + |I \backslash U| \geq r_1(U) + r_2(S \backslash U)$, which establishes that $|I| = r_1(U) + r_2(S \backslash U)$. Therefore, I is a max cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$. □

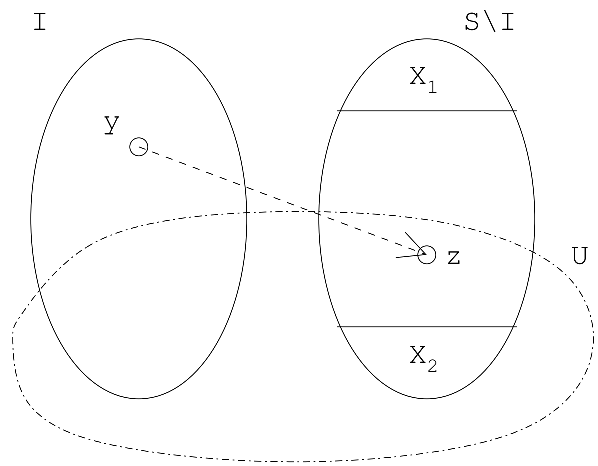

Figure 3: Exchange Graph with a (y, z) arc entering U

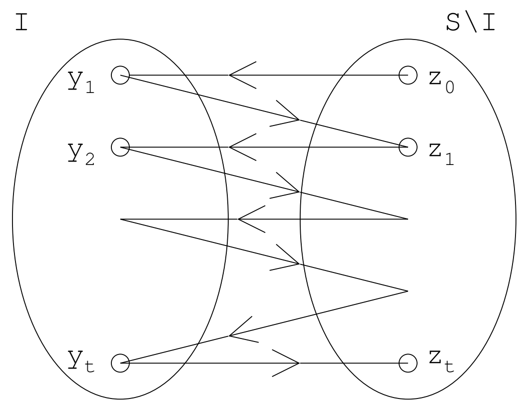

Figure 4: A path P in $D_{M_1,M_2}(I)$

Lemma 7 *If P is a shortest $X_1 - X_2$ path in $D_{M_1,M_2}(I)$, then $I' = I\Delta V(P)$ is in $\mathcal{I}_1 \cap \mathcal{I}_2$.*

Proof: Recall the following lemma from the previous lecture which we will use here:

Lemma 8 *Let $M = (S, \mathcal{I})$ be a matroid. Let $I \in \mathcal{I}$ and $J \subseteq S$ such that $|I| = |J|$. If there is a unique perfect matching on $I\Delta J$ in $A(I)$, then $J \in \mathcal{I}$.*

Let $P = z_0, y_1, z_1, \ldots, y_t, z_t$ (shown in Figure 4) be a shortest path from X_1 to X_2. Let $J = \{z_1, \ldots, z_t\} \cup (I \backslash \{y_1, \ldots, y_t\})$. Then $J \subseteq S$, $|J| = |I|$, and the arcs from $\{y_1, \ldots, y_t\}$ to $\{z_1, \ldots, z_t\}$ form a unique perfect matching from $I \backslash J$ to $J \backslash I$ (otherwise P has a short cut and is not a shortest path). Then by Lemma 8, $J \in \mathcal{I}_1$.

Also, $z_i \notin X_1$ for $i \geq 1$, otherwise P would not be the shortest possible $X_1 - X_2$ path. This implies that $z_i + I \notin \mathcal{I}_1$, which implies that $r_1(I \cup J) = r_1(I) = r_1(J) = |I| = |J|$. Then since $I + z_0 \in \mathcal{I}_1$, it follows that $J + z_0 \in \mathcal{I}_1$ (i.e., $I' = (I \backslash \{y_1, \ldots, y_t\}) \cup \{z_0, z_1, \ldots, z_t\} \in \mathcal{I}_1$).

By symmetry, $I' \in \mathcal{I}_2$. This implies that $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$. $\square$

Theorem 9 *There is a polynomial time algorithm to find a maximum cardinality set in the intersection of two matroids.*

Algorithm 1 will compute a maximum cardinality independent set in the intersection of two matroids M_1 and M_2 in polynomial time. This algorithm can be adapted to find a maximum weight independent set in the intersection of two matroids by adding appropriate weights to the vertices in $D_{M_1,M_2}(I)$ and searching for the shortest weight path with the fewest number of arcs among all such paths of shortest weight.

Algorithm 1 Algorithm for Maximum Cardinality Independent Set in Intersection of Two Matroids

procedure MAXINDEPSET($M_1 = (S, \mathcal{I}_1)$, $M_2 = (S, \mathcal{I}_2)$)
  $I \leftarrow \emptyset$
  repeat
    Construct $D_{M_1,M_2}(I)$
    $X_1 \leftarrow \{z \in S \backslash I \mid I + z \in \mathcal{I}_1\}$
    $X_2 \leftarrow \{z \in S \backslash I \mid I + z \in \mathcal{I}_2\}$
    Let P be a shortest $X_1 - X_2$ path in $D_{M_1,M_2}(I)$
    if P is not empty **then**
      $I \leftarrow I \Delta V(P)$ ▷ $I' = (I \backslash \{y_1, \ldots, y_t\}) \cup \{z_0, z_1, \ldots, z_t\}$
    end if ▷ Else P is empty and I is maximal
  until I is maximal
end procedure