CS 598CSC: Combinatorial Optimization **Lecture date: 03/11/2010**
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1 Matroid Polytope

In the previous lecture we saw that for a matroid $M = (S, \mathcal{I})$ the following system of inequalities determines the convex hull of the independent sets of M (i.e., sets in $\mathcal{I}$):

$$\begin{array}{rcll} x(U) & \leq & r_M(U) & U \subseteq S \\ x(e) & \geq & 0 & e \in S \end{array}$$

where $r_M(\cdot)$ is the rank function of M. The proof was based on a dual fitting technique via the Greedy algorithm for a maximum weight independent set problem.

In this lecture, we will give a different primal proof that is built on uncrossing. This is based on [2].

Theorem 1 *Let x be an extreme point of the polytope*

$$(*) \begin{cases} x(U) \leq r_M(U) & U \subseteq S \\ x(e) \geq 0 & e \in S \end{cases}$$

Then there is some $e \in S$ such that $x(e) \in \{0, 1\}$.

The following corollary follows by induction from Theorem 1. We leave a formal proof as an exercise.

Corollary 2 *The system of inequalities $(*)$ determine the independent set polytope of $M = (S, \mathcal{I})$.*

Now we turn our attention to the proof of Theorem 1.

Proof of Theorem 1. Let x be an extreme solution for the polytope $(*)$. Suppose that M has a loop e. Since $r_M(\{e\}) = 0$, it follows that $x(e) = 0$ and we are done. Therefore we may assume that M does not have any loops and thus the polytope $(*)$ is full dimensional[1]. Now suppose that $x(e) \in (0, 1)$ for all elements $e \in S$. Let n denote the number of elements in S. Let

$$\mathcal{F} = \{U \mid U \subseteq S, x(U) = r_M(U)\}$$

Differently said, $\mathcal{F}$ is the set of all sets whose constraints are tight at x (i.e., sets whose constraints are satisfied with equality by the solution x).

Before proceeding with the proof, we note that the submodularity for the rank function $r_M(\cdot)$ implies that $\mathcal{F}$ has the following "uncrossing" property.

[1] A polytope is full dimensional if it has an interior point, i.e., a point x that does not satisfy any of the constraints with equality. Consider x such that, for all e, $x(e) = \epsilon$ for some $0 < \epsilon < 1/|S|$. Clearly, $x(e) > 0$ for any e. For any set U, we have $x(U) = \epsilon|U| < 1$. If M does not have any loops, $r_M(U) \geq 1$ for all sets U. Thus M is full-dimensional if there are no loops.

Lemma 3 *If $A, B \in \mathcal{F}$ then $A \cap B$ and $A \cup B$ are in $\mathcal{F}$.*

Proof: Let A and B be two sets in $\mathcal{F}$; thus $x(A) = r_M(A)$ and $x(B) = r_M(B)$. It follows from the submodularity of the rank function that

$$x(A) + x(B) = r_M(A) + r_M(B) \geq r_M(A \cap B) + r_M(A \cup B)$$

Additionally,

$$x(A) + x(B) = x(A \cap B) + x(A \cup B)$$

Therefore $x(A \cap B) + x(A \cup B) \geq r_M(A \cap B) + r_M(A \cup B)$. Since $x(A \cap B) \leq r_M(A \cap B)$ and $x(A \cup B) \leq r_M(A \cup B)$, it follows that $x(A \cap B) = r_M(A \cap B)$ and $x(A \cup B) = r_M(A \cup B)$. Thus $A \cap B$ and $A \cup B$ are also in $\mathcal{F}$. $\square$

Let $\chi(U)$ denote the characteristic vector of U. Since x is a vertex solution, $(*)$ is full dimensional, and $x(e) \neq 0$ for all e, there is a collection $\{U_1, U_2, \ldots, U_n\}$ of n sets such that x satisfies the constraint corresponding to each U_i with equality (i.e., $x(U_i) = r_M(U_i)$ for $1 \leq i \leq n$) and the vectors $\chi(U_1), \ldots, \chi(U_n)$ are linearly independent. Therefore the set $\{\chi(U) \mid U \in \mathcal{F}\}$ has n linearly independent vectors.

For a set $\mathcal{A} \subseteq 2^S$, let $span(\mathcal{A})$ denote $span(\{\chi(U) \mid U \in \mathcal{A}\})$, where $\chi(U)$ is the characteristic vector of U.

Lemma 4 *There exists a laminar family $\mathcal{C} \subseteq \mathcal{F}$ such that $span(\mathcal{C}) = span(\mathcal{F})$. Moreover, $\mathcal{C}$ is a **chain**, i.e., for any two sets $A, B \in \mathcal{C}$, either $A \subseteq B$ or $B \subseteq A$.*

Assuming Lemma 4, we can complete the proof of Theorem 1 as follows. Let $\mathcal{C}$ be the chain guaranteed by Lemma 4. Since $span(\mathcal{C}) = span(\mathcal{F})$, there exists a chain $\mathcal{C}' \subseteq \mathcal{C}$ such that $|\mathcal{C}'| = n$ and x is the unique solution to the system

$$x(U) = r_M(U) \qquad U \in \mathcal{C}'$$

Let $\mathcal{C}' = \{A_1, A_2, \ldots, A_n\}$; wlog, $A_1 \subset A_2 \subset \cdots \subset A_n$. Let $A_0 = \emptyset$. Suppose that there exists an i such that $|A_i \setminus A_{i-1}| = 1$, and let $e \in A_i \setminus A_{i-1}$. Now we claim that we must have $x(e) \in \{0, 1\}$. To see why this is true, note that we have

$$x(e) = x(A_i) - x(A_{i-1}) = r_M(A_i) - r_M(A_{i-1})$$

Since $r_M(A_i) - r_M(A_{i-1})$ is an integer and $r_M(A_{i-1}) \leq r_M(A_i) \leq r_M(A_{i-1}) + 1$, it follows that $r_M(A_i) - r_M(A_{i-1}) \in \{0, 1\}$. But this contradicts the fact that $x(e) \in (0, 1)$. Therefore we may assume that $|A_i \setminus A_{i-1}| \geq 2$. But then $|S| \geq 2n$, which is a contradiction.

Finally, we turn our attention to the proof of Lemma 4.

Proof of Lemma 4. Let $\mathcal{C}$ be a chain in $\mathcal{F}$ that is maximal with respect to inclusion (i.e., $\mathcal{C}$ is not a proper subset of any chain in $\mathcal{F}$). We claim that $span(\mathcal{C}) = span(\mathcal{F})$. Suppose not and let $A \in \mathcal{F}$ be such that $\chi(A) \in span(\mathcal{F}) \setminus span(\mathcal{C})$. If there are several such sets A, we choose one that minimizes the number of sets in $\mathcal{C}$ that it *properly* intersects[2].

[2]Two sets X and Y properly intersect if $X \cap Y, X - Y, Y - X$ are all non-empty.

Now suppose that A does not *properly* intersect any set in $\mathcal{C}$. Clearly, $\mathcal{C} + A$ is not a chain, since this contradicts the maximality of $\mathcal{C}$. Therefore there exist $B, B' \in \mathcal{C}$ such that B is the minimal set in $\mathcal{C}$ that contains A and B' is the maximal set in $\mathcal{C}$ that is contained in B. By Lemma 3, $A \cup B'$ is in $\mathcal{F}$. If $A \cup B'$ is a proper subset of B, $\mathcal{C} + (A \cup B')$ is a chain, which contradicts the maximality of $\mathcal{C}$. Therefore we must have $A \cup B' = B$. Since A and B' are disjoint, we have $\chi(A) + \chi(B') = \chi(B)$ and thus $\chi(A)$ is in the span of $\chi(B)$ and $\chi(B')$, which contradicts the fact that $\chi(A) \notin span(\mathcal{C})$.

Therefore we may assume that A properly intersects a set B in $\mathcal{C}$. By Lemma 3, $A \cup B$ and $A \cap B$ are in $\mathcal{F}$.

Proposition 5 *Each of $A \cup B$, $A \cap B$ properly intersects fewer sets in $\mathcal{C}$ than A.*

Assuming Proposition 5, we can complete the proof as follows. It follows from our choice of A that $A \cup B$ and $A \cap B$ are both in $span(\mathcal{C})$. Since $\chi(A) + \chi(B) = \chi(A \cup B) + \chi(A \cap B)$, it follows that $\chi(A)$ is in $span(\mathcal{C})$ as well, which is a contradiction. Therefore it suffices to prove Proposition 5.

Proof of Proposition 5. Since each of $A \cup B$, $A \cap B$ does not properly intersect B, it suffices to show that if a set $B' \in \mathcal{C}$ properly intersects $A \cup B$ (or $A \cap B$) then it properly intersects A as well.

Let $B' \in \mathcal{C}$ be a set that properly intersects $A \cup B$. Since B and B' are both in C, it follows that one of B, B' is a subset of the other. If B' is a subset of B, B' is contained in $A \cup B$ (and thus does not properly intersect $A \cup B$). Therefore B must be a proper subset of B'. Clearly, B' intersects A (since $A \cap B$ is nonempty). If B' does not properly intersect A, it follows that one of A, B' is a subset of the other. If $A \subseteq B'$, it follows that $A \cup B \subseteq B'$, which is a contradiction. Therefore we must have $B \subset B' \subseteq A$, which is a contradiction as well. Thus B' properly intersects A.

Let $B' \in \mathcal{C}$ be a set that properly intersects $A \cap B$. Clearly, B' intersects A and thus it suffices to show that $B' \setminus A$ is nonempty. As before, one of B, B' is a subset of the other. Clearly, B' must be a subset of B (since otherwise $A \cap B \subseteq B \subseteq B'$). Now suppose that $B' \subseteq A$. Since B' is a subset of B, it follows that $B' \subseteq A \cap B$, which is a contradiction. Therefore $B' \setminus A$ is non-empty, as desired. □

2 Facets and Edges of Matroid Polytopes

Recall that the following system of inequalities determines the matroid polytope.

$$(*) \begin{cases} x(U) \leq r_M(U) & U \subseteq S \\ x(e) \geq 0 & e \in S \end{cases}$$

Throughout this section, we assume that the matroid has no loops and thus the polytope is full dimensional.

It is useful to know which inequalities in the above system are redundant. As we will see shortly, for certain matroids, the removal of redundant inequalities gives us a system with only polynomially many constraints.

Recall that a *flat* is a subset $U \subseteq S$ such that $U = span(U)$. Consider a set U that is *not* a flat. Since $r_M(U) = r_M(span(U))$ and $U \subset span(U)$, any solution x that satisfies the constraint

$$x(span(U)) \leq r_M(span(U))$$

also satisfies the inequality

$$x(U) \leq r_M(U)$$

Therefore we can replace the system $(*)$ by

$$(**) \begin{cases} x(F) \leq r_M(F) & F \subseteq S,\ F \text{ is a flat} \\ x(e) \geq 0 & e \in S \end{cases}$$

Definition 1 *A flat F is separable if there exist flats F_1, F_2 such that F_1 and F_2 partition F and*

$$r_M(F_1) + r_M(F_2) = r_M(F)$$

If F is a separable flat, any solution x that satisfies the constraints

$$x(F_1) \leq r_M(F_1)$$

$$x(F_2) \leq r_M(F_2)$$

also satisfies the constraint

$$x(F) \leq r_M(F)$$

since $x(F) = x(F_1) + x(F_2)$ and $r_M(F) = r_M(F_1) + r_M(F_2)$. Therefore we can remove the constraint $x(F) \leq r_M(F)$ from $(**)$. Perhaps surprisingly, the resulting system does not have any redundant constraints. The interested reader can consult Chapter 40 in Schrijver [**?**] for a proof.

Theorem 6 *The system of inequalities*

$$\begin{array}{rcll} x(F) & \leq & r_M(F) & F \subseteq S,\ F \textit{ is an inseparable flat} \\ x(e) & \geq & 0 & e \in S \end{array}$$

is a minimal system for the independent set polytope of a loopless matroid M.

As an example, consider the uniform matroid. The independent set polytope for the uniform matroid is determined by the following constraints:

$$\sum_{e \in S} x(e) \leq k$$

$$x(e) \geq 0 \qquad e \in S$$

Similarly, the independent set polytope for the partition matroid induced by the partition $S_1, \ldots, S_h$ of S and integers $k_1, \ldots, k_h$ is determined by the following constraints:

$$\sum_{e \in S_i} x(e) \leq k_i \qquad 1 \leq i \leq k$$

$$x(e) \geq 0 \qquad e \in S$$

Finally, consider the graphic matroid induced by a graph $G = (V, E)$. The base polytope of a graphic matroid corresponds to the the spanning tree polytope, which is determined by the following constraints:

$$\begin{array}{rcll} x(E[U]) & \leq & |U| - 1 & U \subseteq V \\ x(E) & = & |V| - 1 & \\ x(e) & \geq & 0 & e \in E \end{array}$$

where $E[U]$ is the set of edges inside the vertex set $U \subseteq V$.

Definition 2 *Two vertices x, x' of a polyhedron P are* adjacent *if they are contained in a face F of P of dimension one, i.e., a line.*

Theorem 7 *Let $M = (S, \mathcal{I})$ be a loopless matroid. Let $I, J \in \mathcal{I}$, $I \neq J$. Then $\chi(I)$ and $\chi(J)$ are adjacent vertices of the independent set polytope of M if and only if $|I \triangle J| = 1$ or $|I \setminus J| = |J \setminus I| = 1$ and $r_M(I) = r_M(J) = |I| = |J|$.*

The interested reader can consult Schrijver [**?**] for a proof.

3 Further Base Exchange Properties

We saw earlier the following base exchange lemma.

Lemma 8 *Let B and B' be two bases of a matroid M, and let y be an element of $B' \setminus B$. Then*

(i) there exists $x \in B \setminus B'$ such that $B' - y + x$ is a base

(ii) there exists $x \in B \setminus B'$ such that $B + y - x$ is a base

We will prove a stronger base exchange theorem below and derive some corollaries that will be useful in matroid intersection and union.

Theorem 9 (Strong Base Exchange Theorem) *Let B, B' be two bases of a matroid M. Then for any $x \in B \setminus B'$ there exists an $y \in B' \setminus B$ such that $B - x + y$ and $B' - y + x$ are both bases.*

Proof: Let x be any element in $B \setminus B'$. Since B' is a base, $B' + x$ has a unique circuit C. Then $(B \cup C) - x$ contains a base. Let B'' be a base of $(B \cup C) - x$ that contains $B - x$. We have $B'' = B - x + y$, for some $y \in C - x$.

Now suppose that $B' - y + x$ is not a base. Then $B' - y + x$ has a circuit C'. Since $y \in C \setminus C'$, $B' + x$ has two distinct circuits C, C', which is a contradiction (see Corollary 22 in Lecture 14). Therefore $B' - y + x$ is independent and, since $|B' - y + x| = |B'|$, $B' - y + x$ is a base. $\square$

In fact, Theorem 9 holds when B, B' are independent sets of the same size instead of bases.

Corollary 10 *Let I, J be two independent sets of a matroid $M = (S, \mathcal{I})$ such that $|I| = |J|$. Then for any $x \in I \setminus J$ there exists an $y \in J \setminus I$ such that $I - x + y$ and $J - y + x$ are both independent sets.*

Proof: Let $k = |I| = |J|$. Let $M' = (S, \mathcal{I}')$, where

$$\mathcal{I}' = \{I \mid I \in \mathcal{I} \text{ and } |I| \leq k\}$$

It is straightforward to verify that M' is a matroid as well. Additionally, since every independent set in M' has size at most k, I and J are bases in M'. It follows from Theorem 9 that for any $x \in I \setminus J$ there exists an $y \in J \setminus I$ such that $I - x + y$ and $J - y + x$ are both bases in M', and thus independent sets in M. □

Let $M = (S, \mathcal{I})$ be a matroid, and let $I \in \mathcal{I}$. We define a directed bipartite graph $D_M(I)$ as follows. The graph $D_M(I)$ has vertex set S; more precisely, its bipartition is $(I, S \setminus I)$. There is an edge from $y \in I$ to $z \in S \setminus I$ iff $I - y + z$ is an independent set.

Lemma 11 *Let $M = (S, \mathcal{I})$ be a matroid, and let I, J be two independent sets in M such that $|I| = |J|$. Then $D_M(I)$ has a perfect matching on $I \triangle J$*[3].

Proof: We will prove the lemma using induction on $|I \triangle J|$. If $|I \triangle J| = 0$, the lemma is trivially true. Therefore we may assume that $|I \triangle J| \geq 1$. It follows from Corollary 10 that there exists an $y \in I$ and $z \in J$ such that $I' = I - y + z$ and $J' = J + y - z$ are independent sets. Note that $|I' \triangle J'| < |I \triangle J|$ and $|I'| = |J'|$. It follows by induction that $D_M(I)$ has a perfect matching N on $I' \triangle J'$. Then $N \cup \{(y, z)\}$ is a perfect matching on $I \triangle J$. □

Lemma 12 *Let $M = (S, \mathcal{I})$ be a matroid. Let I be an independent set in M, and let J be a subset of S such that $|I| = |J|$. If $D_M(I)$ has a* unique *perfect matching on $I \triangle J$ then J is an independent set.*

Before proving the lemma, we note the following useful property of unique perfect matchings.

Proposition 13 *Let $G = (X, Y, E)$ be a bipartite graph such that G has a* unique *perfect matching N. Then we can label the vertices of X as $x_1, \ldots, x_t$, and we can label the vertices of Y as $y_1, \ldots, y_t$ such that*

$$N = \{(x_1, y_1), \ldots, (x_t, y_t)\}$$

and $(x_i, y_j) \notin E$ for all i, j such that $i < j$.

Proof: We start by noting that there is an edge $xy \in N$ such that one of x, y has degree one. We construct a trail[4] by alternately taking an edge in N and an edge not in N, until either we cannot extend the trail or we reach a previously visited vertex. Now suppose that the trail has a cycle C. Since G is bipartite, C has even length. Thus we can construct a perfect matching from N by removing the edges of C that are in N and adding the edges of C that are not in N, which contradicts the fact that G has a unique perfect matching. Therefore we may assume that the trail is a path. If the last edge of the trail is not in N, we can extend the trail by taking the edge of N incident to the last vertex. Therefore the last edge must be in N. Then the last vertex on the trail has degree one, since otherwise we could extend the trail using one of the edges incident to it that are not in N. It follows that the last edge of the trail is the desired edge.

Now let xy be an edge in N such that one of its endpoints has degree one in G. Suppose that x has degree one. We let $x_1 = x$, $y_1 = y$, and we remove x and y to get a graph G'. Since $N - xy$

[3] A perfect matching on a set U is a matching such that S is the set of endpoints of the edges in the matching.

[4] A trail is a walk in which all edges are distinct.

is the unique perfect matching in G', it follows by induction that we can label the vertices of G' such that

$$N - xy = \{(x_2, y_2), \ldots, (x_t, y_t)\}$$

such that (x_i, y_j) is not an edge in G', for all $2 \leq i < j \leq t$. Since x_1 has degree one in G, we are done. Therefore we may assume that y has degree one. We let $x_t = x$, $y_t = y$, and we remove x and y to get a graph G'. As before, it follows by induction that we can label the vertices of G' such that

$$N - xy = \{(x_1, y_1), \ldots, (x_{t-1}, y_{t-1})\}$$

such that (x_i, y_j) is not an edge in G', for all $1 \leq i < j \leq t-1$. Since y_t has degree one in G, we are done. $\square$

Proof of Lemma 12. Let G denote the (undirected) subgraph of $D_M(I)$ induced by $I \triangle J$, and let N denote the unique perfect matching in G. Since G is a bipartite graph, it follows from Proposition 13 that we can label the vertices of $I \setminus J$ as $y_1, \ldots, y_t$, and we can label the vertices of $J \setminus I$ as $z_1, \ldots, z_t$ such that

$$N = \{(y_1, z_1), \ldots, (y_t, z_t)\}$$

and $(y_i, z_j) \notin E(G)$, for all $1 \leq i < j \leq t$.

Now suppose that J is not independent, and let C be a circuit in J. Let i be the smallest index such that $z_i \in C$. Consider any element z_j in $C - z_i$. Since $j > i$, it follows that $(y_i, z_j) \notin D_M(I)$. Therefore any element z in $C - z_i$ is in $span_M(I - y_i)$, since for any $z \in C - z_i$, either z is in $I \cap J$ or $z = z_j$ for some j. Hence $C - z_i$ is a subset of $span(I - y_i)$. Since C is a circuit,

$$C \subseteq span(C - z_i) \subseteq span(I - y_i)$$

Thus $z_i \in span(I - y_i)$, which contradicts the fact that $I - y_i + z_i$ is independent. $\square$

Corollary 14 *Let $M = (S, \mathcal{I})$ be a matroid, and let $I \in \mathcal{I}$. Let J be a subset of S with the following properties:*

(i) $|I| = |J|$

(ii) $r_M(I \cup J) = |I|$

(iii) $D_M(I)$ *has a unique perfect matching on* $I \triangle J$

Let e be any element not in $I \cup J$ such that $I + e \in \mathcal{I}$. Then $J + e \in \mathcal{I}$.

Proof: It follows from Lemma 12 that J is independent. Since $r_M(I \cup J) = |I|$, both I and J are maximal independent sets in $I \cup J$. Thus $I \subseteq span(J)$ and $J \subseteq span(I)$. Since $I + e$ is independent, $e \notin span(I)$. As we have seen in Lecture 14, since $J \subseteq span(I)$, it follows that $span(J) \subseteq span(I)$. Therefore $e \notin span(J)$ and thus $J + e$ is independent. $\square$

References

[1] Alexander Schrijver. Combinatorial Optimization: Polyhedra and Efficiency, Chapters 39-40, Vol. B, Springer-Verlag 2003.

[2] L. C. Lau, R. Ravi and M. Singh. Iterative Methods in Combinatorial Optimization. Draft of upcoming book, March 2009.