FINDING THE MINIMUM DISTANCE BETWEEN TWO CONVEX POLYGONS *

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1. Introduction

Suppose that a provisional path of coordinated motion has been planned for two rigid polyhedral bodies \( B_1 \) and \( B_2 \) in 3-space. Then there will exist functions \( R_1(t), R_2(t), x_1(t), x_2(t) \) of a parameter \( t \) designating time such that the set of points occupied by \( B_j \) at time \( t \) is \( R_j(t)B_j + x_j(t) \). To verify the validity of the proposed motion, one can proceed as follows: find the minimum distance \( \delta \) between the convex sets \( R_1(t)B_1 + x_1(t) \) and \( R_2(t)B_2 + x_2(t) \) for \( t = 0 \). Let \( L \) be the diameter of the set \( B_2 \), and from the known form of the functions \( R_1, R_2, x_1, \) and \( x_2 \) find an \( \epsilon \) sufficiently small so that

\[
\left| R_1^{-1}(t)R_2(t) - R_1^{-1}(0)R_2(0) \right| \leq \frac{1}{2} L + \left| x_1(t) + x_2(t) \right| - x_1(0) - x_2(0) \leq \delta,
\]

for all \( 0 \leq t \leq \epsilon \). Then a collision between the moving bodies is impossible for this range of \( t \); hence we can advance \( t \) from 0 to \( \epsilon \), and repeat this step. When successive steps of this kind have brought us from \( t = 0 \) to some final value \( t^* \), we can be sure that the planned path is collision-free.

To use this technique effectively, we need a fast algorithm for estimating the minimum distance between two polyhedra. The present note will address the problem of finding this distance, but only under two drastic simplifying assumptions, namely

(i) \( B_1 \) and \( B_2 \) are assumed to be convex;
(ii) \( B_1 \) and \( B_2 \) are assumed to be two dimensional.

Assuming that \( B_1 \) and \( B_2 \) have a total of \( N \) vertices and are described by clockwise bounding segment lists of the standard kind, an \( O(\log^3 N) \) algorithm for determining the minimum distance between \( B_1 \) and \( B_2 \) will be given. The related problem of finding the minimum distance between a variable point \( x \) and a fixed convex body \( B \) is considered in [2], where an \( O(\log N) \) algorithm is given.

2. The algorithm

Hence let \( B_1 \) and \( B_2 \) be convex polygons. Write

\[ B_1 \pm B_2 = \{ x \pm y : x \in B_1, y \in B_2 \} \]

and

\[-B = \{ -x : x \in B \}, \]

so that \( B_1 + B_2 \) is the so-called Minkowski Sum of \( B_1 \) and \( B_2 \), and \( B_1 - B_2 = B_1 + (-B_2) \). Our problem is to estimate the distance between the point \( x = 0 \) and the set \( B_1 - B_2 \); replacing \( B_2 \) by \(-B_2\), it becomes that of estimating the distance from the origin to the convex set \( B_1 + B_2 \). A fast \( O(N) \) procedure for finding \( B_1 + B_2 \) given \( B_1 \) and \( B_2 \) is described in [1], and is as follows:

(a) The sides of \( B_1 \) and \( B_2 \) are available as circular lists arranged in increasing order of the angle \( \theta \) that each side makes with the \( x \) axis. Merge these two lists
into a single similarly ordered list L.

(b) If a side S of B₂ (resp. B₁) lies between two successive sides S', S'' of B₁ (resp. B₂) in this list, let c be the corner at which S and S' meet. Then S + c is a side of B₁ + B₂, and L lists these sides of B₁ + B₂ in their standard circular order. We will say in what follows that the side S + c of B₁ + B₂ comes from S (which is a side either of B₁ or of B₂), and that c is the corner of B₁ (resp. B₂) that matches the side S of B₂ (resp. B₁).

Our algorithm will apply this construction, but to attain $O(\log^2 N)$ performance will avoid forming the full list L. We proceed as follows.

(i) The sides of any convex polygon B can be oriented so that the interior of B lies to their left. Oriented in this way, the sides of B fall into two (circularly) contiguous groups, one consisting of (vertically) ascending, the other of descending edges. It is clear that the ascending (resp. descending) edges of B₁ + B₂ come from the separate ascending (resp. descending) edges of B₁ and of B₂.

(ii) Let a point x in the plane be given; we wish to determine whether it belongs to B₁ + B₂, and if not, to find its minimum distance to B₁ + B₂. First determine the highest (resp. lowest) corner H₁, L₁ of each B; this can be done in time $O(\log N)$ by binary search of the sides of B₁. Then the highest and lowest corners of B₁ + B₂ are H₁ + H₂ and L₁ + L₂ respectively. If x does not lie in the vertical range bracketed by these two points, it is definitely not in B₁ + B₂. If x does lie in this vertical range, then a horizontal line drawn through x will intersect exactly two sides of B₁ + B₂, one an ascending, the other a descending side, and x belongs to B₁ + B₂ if and only if it lies between these two sides. To find these two intersections, we can start at the lowest corner L₁ + L₂ of B₁ + B₂, and perform a binary search of its ascending (resp. descending) sides. This can be done without actually forming the full collection of its sides, using a technique which we will now explain.

(iii) Let the ascending sides of B₁ + B₂ be enumerated, in bottom-to-top order, as S₁, S₂, ..., Sₙ. At any moment during a binary search of these sides, we will be examining two of these sides Sᵢ, Sᵢ₊₁, and will need to examine a side Sⱼ lying between them. We can suppose that the side S' of B₁ (resp. B₂) from which each side Sⱼ under examination comes is known, and that the corner c of B₂ (resp. B₁) matching S' is also known. For our binary search to retain logarithmic efficiency, we must be able to locate a side Sₖ between Sᵢ and Sⱼ such that k is at least (resp. most) a fixed fraction $\alpha$ (resp. $1 - \alpha$) of the distance from i to j. This can be done as follows. Let Sᵢ come from a side S'(of B₁ or B₂) numbered i', matched by a corner c'' (of B₂ or B₁) whose entering edge is numbered i'', and define $S''$, $j'$ and $j''$, similarly from Sⱼ. If S' and S'' are sides of the same polygon (B₁ or B₂), put

$\Delta' = j' - i'$, $\Delta'' = j'' - i''$.

On the other hand, if S' and S'' are sides of different polygons, put

$\Delta' = j'' - i'$, $\Delta'' = j' - i''$.

In order to avoid detailed enumeration of tediously many cases, we will suppose that S' and S'' are sides of different polygons; the treatment of the cases thereby ignored and of this case are similar.

If $\Delta' > \Delta''$, advance from side i' (of the polygon having S' as a side) halfway toward side j'' of this polygon. Let the side in this intermediate position be T, let its index be m, and find its matching corner c. Then T + c is a side of B₁ + B₂; its index as a side of B₁ + B₂ exceeds i by at least $\frac{1}{2} \Delta'$ and by at most $\frac{1}{2} \Delta' + \Delta''$. Hence m lies at least $\frac{1}{4}$ and at most $\frac{3}{4}$ of the way from i to j.

Similarly, if $\Delta'' > \Delta'$, advance from side i'' (of the polygon having c'' as a corner) halfway toward side i' of this polygon. Let the side in this intermediate position be T, its index be m, and its matching corner be c. Then again T + c is a side of B₁ + B₂ whose index is at least $\frac{3}{4}$ and at most $\frac{1}{4}$ of the way from i to j.

Locating the corner matching a given side can be done in time $O(\log N)$, so overall the binary search we have just described requires $O(\log^2 N)$ time.

(iv) The binary search will locate the two points of intersection of the horizontal line through x with the boundary of B₁ + B₂. If x lies between these points it is interior to B₁ + B₂, and we are finished. Otherwise x lies to the right or to the left of one of them. Suppose, for the sake of definiteness, that x lies to the right of B₁ + B₂, or, if x lies above (resp. below) the topmost (resp. bottom most) point of B₁ + B₂, that it lies to the left of this point. Then the point of B₁ + B₂ lying closest to x lies on one of the ascending edges forming the left-hand part Q of the boundary of B₁ + B₂. We now begin to search for this edge. We start this search.
from an edge of $B_1 + B_2$ visible from $x$. Such an edge is available in all cases, since if $x$ lies above (resp. below) $B_1 + B_2$ we have only to take the topmost (resp. bottom most) edge of $Q$.

(v) To find the edge $S^*$ of $B_j + B_2$ containing the point $Z$ closest to $x$, we start with an edge $S$ visible from $x$ and draw a line from $x$ to the initial corner $c_1$ of $S$ (note again that the edges of $Q$ are oriented and point downward). Let $C_2$ be the other corner of $S$. If the angle $xc_1c_2$ is acute, then $N$ lies on $Q$ below $c_1$; if obtuse, then at or above $c_1$. This observation enables the edge $S$ containing $Z$ to be located by binary search. As previously, this binary search procedure will run in time $O(\log_2 N)$. Suppose now that $S$ contains $Z$. Then $xc_1c_2$ is acute but $xc_2c_1$ is obtuse $Z$ is $c_2$; if $xc_1c_2$ is obtuse then $Z$ is $c_1$; and otherwise $Z$ is the foot of the perpendicular from $x$ to $S$.

3. A technique for accelerating the expected speed of location of a point on a divided real axis

Like many other geometric algorithms, the algorithm sketched in the preceding pages makes repeated use of the following computational step:

Given a fixed increasing sequence of real numbers $x_1, \ldots, x_n$, and a point $x$, locate the interval $(x_i, x_{i+1})$ in which $x$ lies.

The normal technique for accomplishing this is simply to perform a binary search, which requires time $O(\log n)$.

We will now sketch an alternative approach which has the same worst case behavior, but (if the points $x_j$ are randomly distributed) will reduce the expected time needed to locate the desired interval to $O(1)$.

This is simply to keep an auxiliary table $T$ consisting of $\frac{n}{2}$ locations. To set up $T$, we divide the full range $(x_1, x_n)$ from the minimum to the maximum of the $x_i$ into an equal subintervals $I$, each of which corresponds to an entry $E$ of $T$; $E$ then stores the indices of the largest and smallest $x_j$ belonging to $I$. To find the

<table>
<thead>
<tr>
<th>Value of $\alpha$</th>
<th>Value of $e^{-\alpha} \sum_{j \geq 2} \alpha^j \log j/j!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.22</td>
</tr>
<tr>
<td>2</td>
<td>0.57</td>
</tr>
<tr>
<td>4</td>
<td>1.20</td>
</tr>
</tbody>
</table>

Table 1 shows this last function.

This technique can be used in the 'find matching corner' step of the closest point algorithm sketched earlier, and, assuming a random distribution of angles parallel to the sides of the polygons involved, will reduce the expected time needed to find this corner to $O(1)$; thus the expected time required for the whole algorithm is $O(\log N)$ rather than $O(\log^2 N)$.

References
